A Time Continuous Analysis of the Primary and Secondary Bond Markets

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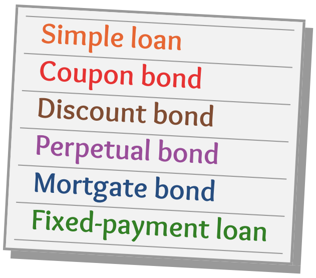

There exist two credit markets in which bonds are bought and sold.[1] They are the primary bond market and the secondary bond market. In the primary bond market, the purchaser of a bond effectively lends money to the bond issuer, i.e., the borrower. Both the purchaser and the issuer benefit from this transaction. The purchaser receives (possibly nil) regular payments over the agreed period, together with a (possibly nil) lump sum payment at the end of the period. And the issuer receives an immediate source of funds to put to good use. In the secondary bond market, the purchaser buys the rights to an existing bond, which include both the abovementioned payment terms together with any risk of default.

There is therefore conceptually little difference between the primary and the secondary markets, except that it is in the primary market that the bonds are brought into existence by virtue of the issuer requiring an immediate source of funds.

In this article, a quantitative analysis of bond trading is given. Two generic time continuous bond pricing equations are formally derived, one for the primary bond market, and one for the secondary bond market. Five types of bonds are identified via specific application of these equations. The well-known qualitative reciprocity between a coupon bond's price and prevailing bond interest rates is placed on a firm quantitative footing. Finally, the evolution of a bond investor's relative return is studied, and the impact of interest rate fluctuations on the investor's return is revealed.

1 Primary bond market

A generic time continuous equation which may be used to govern bond transactions in the primary bond market is:

$$\boxed{B(t) = \left(B_0 - \int_{t_0}^{t} \eta(\tau) e^{-\int_{t_0}^{\tau} \beta(\tau')\,\mathrm{d}\tau'}\,\mathrm{d}\tau \right) e^{\int_{t_0}^{t} \beta(\tau)\,\mathrm{d}\tau}} \tag{1}$$

where $B(t)$ is the value of the bond at some time t subsequent to the purchaser having bought the bond at time[2] t_0; B_0 is the amount the purchaser paid initially, i.e., the amount being lent to the borrower, who is also known as the bond issuer; $\eta(t)$ is the payment rate at time t at which the bond issuer must pay the bond purchaser; and $\beta(t)$ is the agreed time rate of the bond interest at time t. The bond interest rate $\beta(t)$ is also termed the "yield to maturity."

[1] This work derives in part from a desire to better understand the reciprocal relationship which exists in the secondary bond market between a bond's price and the prevailing bond interest rate. This work is entirely my own, and reflects my current understanding. It has not been peer reviewed.

[2] The choice of the unit of continuous time is arbitrary. Typical units are days or months.

Equation (1) is easily derived, as follows. The infinitesimal change in the value of the bond at any time t must be:

$$\mathrm{d}B(t) = B(t)\beta(t)\,\mathrm{d}t - \mathrm{d}P(t)$$

where $P(t)$ is the total payment made by the borrower to the lender since inception, i.e., over the time interval $[t_0, t]$. As the interest term, first term above increases the infinitesimal change in the value of the bond. And as the payment term, the second term decreases it. Since $\eta(t)$ is the payment rate, we have:

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = \eta(t)$$

The equation governing the primary bond is therefore:

$$\frac{\mathrm{d}B(t)}{\mathrm{d}t} - \beta(t)B(t) = -\eta(t) \tag{2}$$

This is a linear homogenous differential equation. To obtain an exact solution for $B(t)$, we first seek some function $g(t)$ which satisfies:

$$g\frac{\mathrm{d}B}{\mathrm{d}t} - g\beta B = g\frac{\mathrm{d}B}{\mathrm{d}t} + \frac{\mathrm{d}g}{\mathrm{d}t}B = \frac{\mathrm{d}}{\mathrm{d}t}(gB)$$

It is satisfied provided that

$$\begin{aligned}
&\frac{\mathrm{d}g}{\mathrm{d}t} = -g\beta \\
\Rightarrow\ &\int_{g(t_0)}^{g(t)} \frac{\mathrm{d}\rho}{\rho} = -\int_{t_0}^{t} \beta(\tau)\,\mathrm{d}\tau \\
\Rightarrow\ &\ln\left(\frac{g(t)}{g(t_0)}\right) = -\int_{t_0}^{t} \beta(\tau)\,\mathrm{d}\tau \\
\Rightarrow\ &g(t) = g(t_0)e^{-\int_{t_0}^{t} \beta(\tau)\,\mathrm{d}\tau}
\end{aligned}$$

Multiplying both sides of (2) by this $g(t)$ gives:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(gB) = -g\eta \\
\Rightarrow\ &g(t)B(t) - g(t_0)B(t_0) = -\int_{t_0}^{t} g(\tau)\eta(\tau)\,\mathrm{d}\tau \\
\Rightarrow\ &B(t) = \frac{g(t_0)}{g(t)}B(t_0) - \frac{1}{g(t)}\int_{t_0}^{t} g(\tau)\eta(\tau)\,\mathrm{d}\tau
\end{aligned}$$

But since $\dfrac{g(t_0)}{g(t)} = e^{\int_{t_0}^{t} \beta(\tau)\,\mathrm{d}\tau}$, it is apparent that the initial condition $B(t) = B_0$ is satisfied for any $g(t_0)$. And so we are free to set $g(t_0) = 1$. We thus obtain the exact solution above in (1).

For most bonds the interest rate and the payment rate are fixed when the bond is issued. So if we restrict our analysis to such bonds, then $\eta(t) = \eta$ and $\beta(t) = \beta$ for all t, and (1) simplifies to:

$$B(t) = \left(B_0 - \eta\int_{t_0}^{t} e^{-\beta(\tau - t_0)}\,\mathrm{d}\tau\right)e^{\beta(t-t_0)}$$

The solution is

$$\boxed{B(t) = B_0 e^{\beta(t-t_0)} + \frac{\eta}{\beta}\left(1 - e^{\beta(t-t_0)}\right)} \tag{3}$$

The first term in (3) works to increase the size of the bond over time. Indeed, the size would grow exponentially without limit if the borrower's payment rate η is zero. The second term works to decrease the size of the bond over time.

2 Types of bonds

An application of (3) identifies five types of bond instruments available in credit markets, namely, *simple loans*, *fixed-payment loans*, *coupon bonds*, *discount bonds* (also known as zero-coupon bonds), and *perpetual bonds*.

Simple loans. A simple loan is one in which the bond issuer makes no payments $\mathrm{d}P(t)$, but does pay interest. Setting the payment rate $\eta = 0$ in (3) results in:

$$B(t) = B_0 e^{\beta(t-t_0)} \tag{4}$$

The "money market" is an example of a market in which simple loans are applied. By depositing funds into a money market account, an investor is actually lending an amount B_0 to the bank. The bank simply applies (4). And although the investor receives no payment stream, the investor's funds grow according to (4) until the amount $B(T)$ is withdrawn from the account at maturity time T.

Fixed-payment loans. A fixed-payment loan is one in which the bond issuer commits to a payment stream at fixed payment rate η, but does not imburse the bond purchaser with a final lump sum at maturity. So setting $B(T) = 0$ in (3) for some time $T > t_0$ at maturity results in:

$$B(T) = B_0 e^{\beta(T-t_0)} + \frac{\eta}{\beta}\left(1 - e^{\beta(T-t_0)}\right) = 0$$

This sets the prescription for the payment rate η as a function of the time lapse $T - t_0$ till maturity, the bond fixed interest rate β, and the initial bond purchase value B_0, as

$$\eta = \frac{e^{\beta(T-t_0)}}{e^{\beta(T-t_0)} - 1}\beta B_0$$

The mortgage bond market is an example in which fixed payment stream loans are applied. A lending institution lends an amount B_0 to a mortgagee. In return, the mortgagee is required to pay at a rate η over the time interval $[t_0, T]$, after which the mortgagee owes nothing more ($B(T) = 0$).

Coupon bonds. A coupon bonds is one in which the bond issuer commits to a payment stream at fixed payment rate η, and at maturity reimburses the bond purchaser the initial borrowed amount. Therefore, setting $B(T) = B(t_0)$ in (3) results in:

$$\eta = \beta B_0 \tag{5}$$

The primary and secondary bond markets are examples in which coupon bonds are applied. By investing an amount B_0, the investor is rewarded with a payment stream η followed by the refund of the "principle" amount B_0.

Discount bonds (or zero-coupon bonds). Setting $\eta = 0$ in (3) results in:

$$B(t) = B_0 e^{\beta(t-t_0)}$$

Since the value of the bond at the time T at maturity is $B(T)$, the initial value of the bond may be expressed as a discount of $B(T)$:

$$B_0 = \beta_\mathsf{D} B(T)$$

where $\beta_\mathsf{D} \equiv e^{-\beta(T-t_0)}$ is termed the "discount rate." It is immediately evident that discount bonds are identical to simple loans. The difference is merely a semantic one, I think. Whereas simple loans emphasise the initial purchase price B_0, discount bonds emphasise the future price at maturity and express the initial price as some discount of the latter.

Perpetual bonds. Perpetual bonds never mature. What payment stream would the bond issuer commit to under a regime in which his bond issuance never matures? Rewriting (3) as

$$\eta = \beta B_0 + (\eta - \beta B(t))\, e^{-\beta(t-t_0)}$$

it is easy to see that $\eta \to \beta B_0$ as $t \to \infty$ so that in the limit

$$\eta = \beta B_0 \tag{6}$$

Perpetual bonds are therefore never matured, and the bond purchaser receives a payment stream at the agreed payment rate (6) rate forever.

3 Secondary bond market

Bond pricing. Of course, (1) and (3) are well and good provided that the initial bond transaction is the only transaction to take place over the time interval $[t_0, T]$. This is very unlikely. Bonds are frequently bought and sold subsequent to their initial issuance in the primary market. And it is in the secondary market that such transactions occur.

To understand how bonds are priced in the secondary market we must appeal to (1) or (3) even though they apply ostensibly to the primary market. Writing B_0 as the subject in (3) and applied at time T at maturity gives:

$$B_0 = B(T)e^{-\beta(T-t_0)} + \frac{\eta}{\beta}\left(1 - e^{-\beta(T-t_0)}\right) \tag{7}$$

Equation (7) may be considered a prescription for continuous pricing in the primary bond market. Should an investor wish to receive a payment stream at rate η over time interval $[t_0, T]$, followed by a lump sum payment $B(T)$, then the investor may pay (i.e., lend) a bond issuer an amount B_0 now. Once the investor has paid the bond issuer the amount B_0, the investor can be thought of as owning the triple $(B(T), T - t_0, \eta)$—a triple which becomes something sellable in the secondary market.

Sellable, but at what price? Suppose that at some time $t < T$ since having purchased the bond, the bond holder wishes to sell his $(B(T), T - t_0, \eta)$ bond triple. To a potential buyer, however, the relevant triple is instead $(B(T), T - t, \eta)$ because at time t the bond will be maturing after a time lapse equal to $T - t$, not $T - t_0$. Furthermore, there must surely be other bonds available in the primary market at time t which will also mature after the same time lapse $T - t$. By implication, as time progresses, the holder's bond, which has a maturity $T - t_0$, will be competing with other bonds of progressively shorter maturities. Therefore, if the holder's bond is to be sold competitively at time t, it would have to be sold at a price given by

$$\boxed{B(T,t) = B(T,T)e^{-\beta(t)(T-t)} + \frac{\eta}{\beta(t)}\left(1 - e^{-\beta(t)(T-t)}\right)} \tag{8}$$

where the notation "$B(T,t)$" now refers to a price for the original bond with maturity value $B(T,T)$; $\beta(t)$ is the prevailing interest rate fixed at time t, notably now time dependent; and $B(T,T)$ corresponds to the $B(T)$ in (3) and (7). By setting $t = t_0$ and $t = T$, it is easy to see that (8) is a natural generalisation of (7).

An important observation is that the payment rate η in (8) does not depend on the time t. This is because, being a member of the bond holder's sellable triple $(B(T), T - t_0, \eta)$, its value was determined contractually when the holder first purchased the bond at time t_0.

4 Return on investment

What is the original bond holder's return on investment after selling his triple $(B(T,T), T - t_0, \eta)$ at time t in the secondary bond market? The holder purchased the bond at time t_0 at a price $B(T, t_0)$. And since having received a payment stream over the time interval $[t_0, t]$ totalling $P(t)$, his return must be

$$\Delta W(T,t) = P(t) + (B(T,t) - B(T,t_0))$$

The equation together with (8) may be somewhat daunting to analyse. But if we limit attention to the special case of coupon bonds, then using (5) and the fact that $B(T,T) \equiv B(T,t_0)$ for coupon bonds, a sensible prediction for the bond's value at time t becomes

$$\boxed{B(T,t) = B(T,t_0)\left[\frac{\beta(t_0)}{\beta(t)} + \left(1 - \frac{\beta(t_0)}{\beta(t)}\right) e^{-\beta(t)(T-t)}\right]} \tag{9}$$

The tendency for bond pricing $B(T,t)$ in the secondary market to vary reciprocally with variation in the prevailing bond rate $\beta(t)$ is quantitatively revealed in (9).

Relative return. Since the bond holder has (hopefully) been receiving a payment stream over the $[t_0, t]$ interval at a rate $\eta(t)$, the total payment received must be

$$P(t) = \int_{t_0}^{t} \eta(t)\,\mathrm{d}t$$

And once again, assuming the special case of coupon bonds, using (5)

$$P(t) = \int_{t_0}^{t} \beta(t_0)B(T,t_0)\,\mathrm{d}t = \beta(t_0)B(T,t_0)(t - t_0)$$

A sensible prediction for the holder's relative return after selling his coupon bond investment at time t is therefore

$$\boxed{\Delta w(T,t) \equiv \frac{\Delta W(T,t)}{B(T,t_0)} = \beta(t_0)(t-t_0) + \left(\frac{\beta(t_0) - \beta(t)}{\beta(t)}\right)\left(1 - e^{-\beta(t)(T-t)}\right)} \tag{10}$$

Rising bond interest rates. Equation (10) offers a prediction for the evolution of an investor's future relative return as a function of future prevailing bond interest rates $\beta(t)$ and the bond's future time lapse till maturity $T - t$.

It is an important result in that it exposes the risk of loss to the bond investor under a regime of rising interest rates. Suppose that rates have been declining, but that at time t_0 the decline stops at a minimum value of β_0, say. If we may approximate that over the time interval $[t_0, t_1]$, the rate will increase sinusoidally to a maximum β_1, then with some simple algebra it is easy to show that we may write for $\beta(t)$:

$$\beta(t) = \tfrac{1}{2}(\beta_1 + \beta_0) - \tfrac{1}{2}(\beta_1 - \beta_0)\,\sin\left(\frac{2t + t_1 - 3t_0}{t_1 - t_0}\cdot\frac{\pi}{2}\right) \tag{11}$$

Of course, real fluctuations of bond interest rates are more erratic than those captured in this sinusoidal approximation. But the approximation does allow for analysis of predicted returns amidst certain fluctuations. For instance, scrutiny of the South African 10-year government bond rate history from 2000 to 2010 confirms that over a two-year period the difference $|\beta_1 - \beta_0|$ can be up to 3%.[3]

In early 2007, the rate increased from a minimum of about 0.075 per annum, and peaked at 0.1 per annum by mid-2008. An application of the sinusoidal approximation gave a predicted evolution of relative return over this period as shown in Figure 1. At the end of the $1\frac{1}{2}$–year period, the investor's relative return was calculated to be -0.03 despite the investor having received an interest payment stream at a rate 0.075 per annum over this period.

Falling bond interest rates. It is worth contrasting the detriment to the investor of a rising interest rate, as shown in Figure 1, with the benefit of a falling rate. Suppose that at the beginning of the same $1\frac{1}{2}$–year period the rate peaked at 0.1 per annum, and at the end of the period it slumped to 0.075 per annum. An application, once again, of the sinusoidal approximation is shown in Figure 2. At the end of the $1\frac{1}{2}$–year period, the investor's relative return is 0.3. These observations are summarised in Figure 3.

[3] Ref: www.tradingeconomics.com

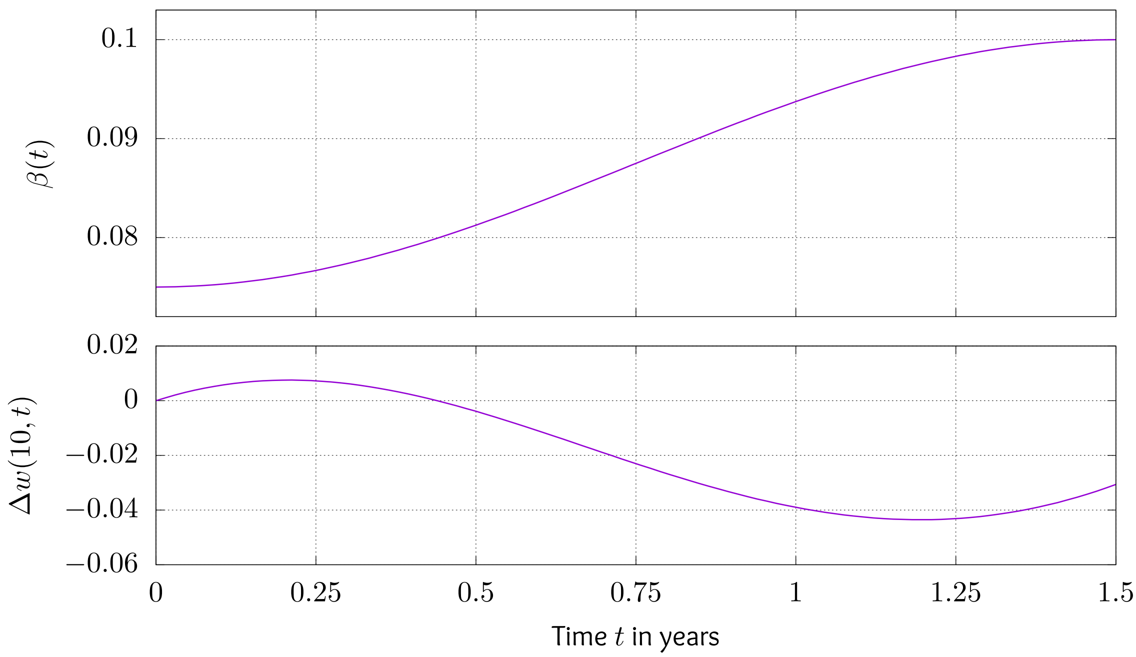

Figure 1: A rising interest rate regime. *Above*—Evolution of a future bond interest rate $\beta(t)$ modelled by equation (11). *Below*—Future relative return $\Delta w(T,t)$ predicted by equation (10), assuming a time lapse till maturity of $T = 10$ years and a current time $t_0 = 0$, with $\beta(t_0 = 0) = \beta_0 = 0.075$ per annum and $\beta(t_1 = 1.5) = \beta_1 = 0.1$ per annum.

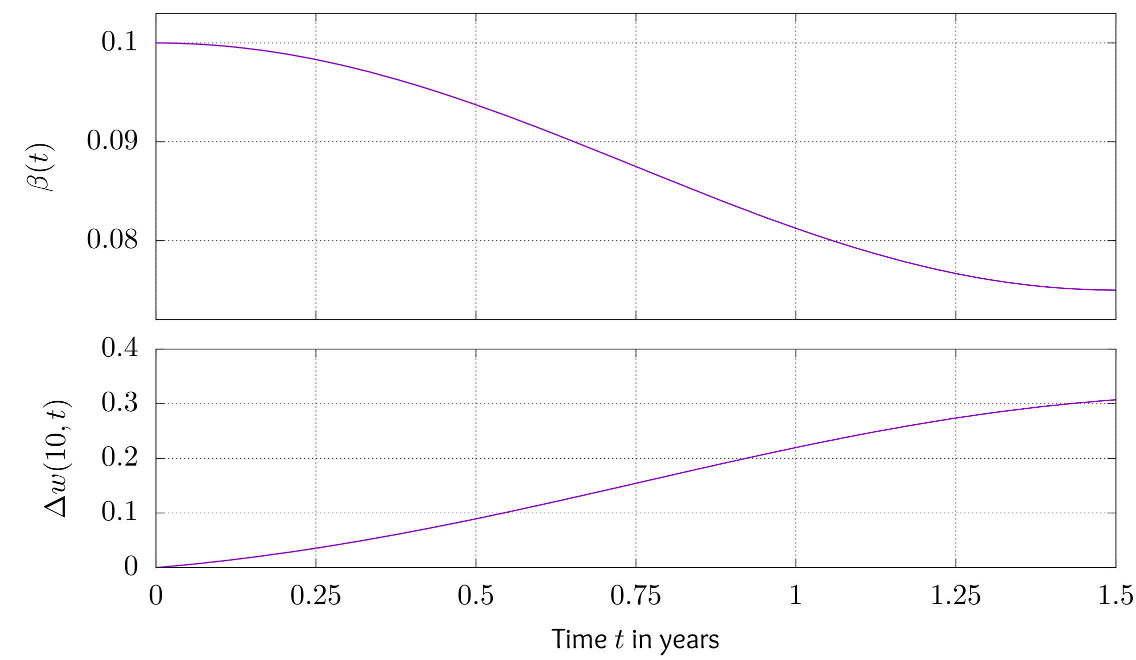

Figure 2: A falling interest rate regime. *Above*—Evolution of a future bond interest rate $\beta(t)$ modelled by equation (11). *Below*—Future relative return $\Delta w(T,t)$ predicted by equation (10), assuming a time lapse till maturity of $T = 10$ years and a current time of $t_0 = 0$, with $\beta(t_0 = 0) = \beta_0 = 0.1$ per annum and $\beta(t_1 = 1.5) = \beta_1 = 0.075$ per annum.

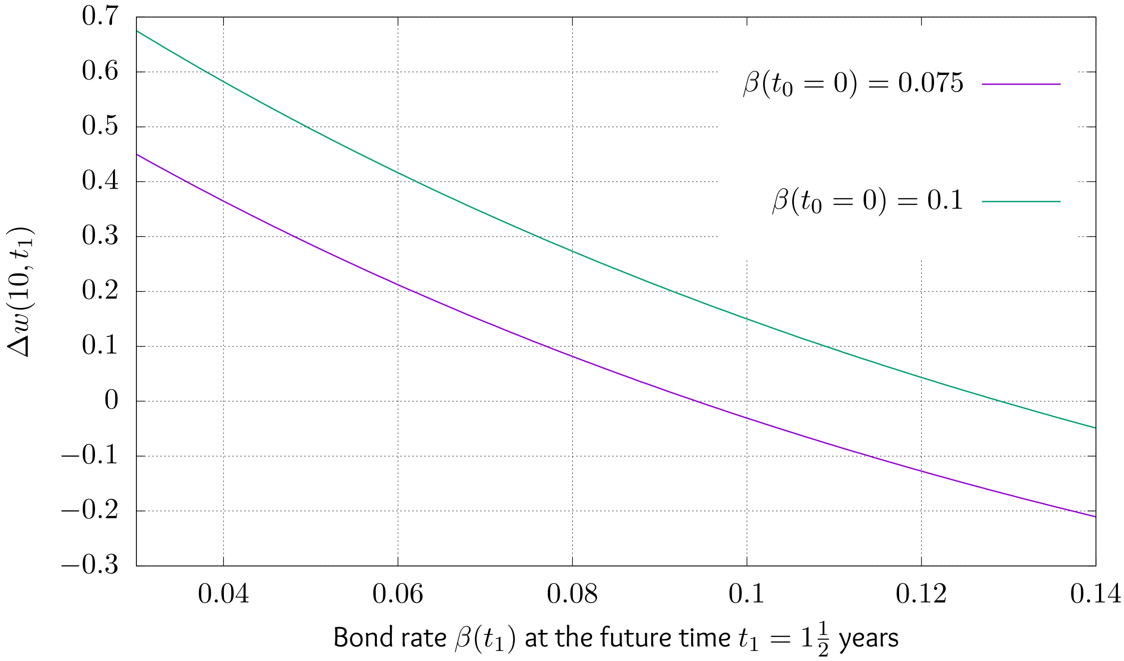

Figure 3: Plot of the future relative return $\Delta w(T, t_1)$ as a function of $\beta(t_1)$ with $t_1 = 1.5$ years. Computed using equation (10) and assuming a time lapse till maturity of $T = 10$ years.