Inflation Adjusted Pricing and the US Gross Domestic Product

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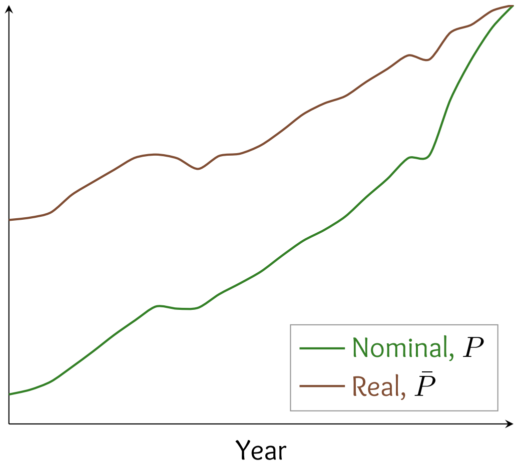

INFLATION OBSCURES the intrinsic value of products and services available in an economy. This is because in an unregulated economy, prices of goods and services are allowed to vary over time. So to better understand the actions of the agents in an economy, the effects of inflation must be accounted for. Doing so impels a distinction between the notions of *real pricing* and *nominal pricing*. In this article, I derive formulae for forward real pricing and backward real pricing over time periods in which inflation and savings may vary. Along the way, I derive the well-known Fisher Equation from first principles. In financial mathematics, the Fisher Equation expresses the relationship between nominal interest rates, real interest rates, and inflation. Finally, I use the analysis presented herein to calculate histories of the US real, nominal, and real per-capita Gross Domestic Products.

The analysis here is time-discrete rather than time-continuous. This is because when central state statisticians release values for the inflation rate, they assume discrete time.

Forward real pricing.

I begin by stating the obvious. To say that at time t_J we have a monetary amount P_J means that at t_J we have the ability to exchange P_J units of currency for N_J products or services, say. This is because, taken together, a supplier of those products or services has deemed them to be of equal value to P_J. Stated reciprocally, at t_J we are able to exchange a single unit of our currency for N_J/P_J units of product or service.

Purchasing ability. But suppose that instead of making the exchange of currency for products or services at t_J, we place P_J in an investment which offers a *nominal return* over the $(t_J, t_{J+1}]$ time interval such when we arrive at time t_{J+1}, we have a nominal monetary amount P_{J+1}.

But unfortunately for us as holders of P_{J+1}, the supplier has now deemed that at t_{J+1} the N_J products or services are no longer worth P_J, but rather $(1 + \alpha_{J+1}\Delta t_{J+1})P_J$, where α_{J+1} is the *time rate of relative increase in the monetary value* of the N_J items at t_{J+1}, i.e., the inflation rate, and where

$$\Delta t_{J+1} \equiv t_{J+1} - t_J$$

So at t_{J+1}, to purchase the N_J items from the supplier, we would have to dispense with $(1 + \alpha_{J+1}\Delta t_{J+1})P_J$ units of our currency. Or stated reciprocally, at t_{J+1}, a single unit of our currency would now be able to purchase only $N_J/[(1+\alpha_{J+1}\Delta t_{J+1})P_J]$ items. But since at t_{J+1} we now have P_{J+1} units of currency, we are in fact able to purchase

$$N_{J+1} = \frac{N_J}{(1+\alpha_{J+1}\Delta t_{J+1})P_J} P_{J+1}$$

units of product or service.

It is therefore clear that the *real relative return* of the investment over the $(t_J, t_{J+1}]$ time interval is not the relative change in our nominal monetary value. Instead, it must be the *relative change in our purchasing ability*, namely:

$$\frac{N_{J+1}-N_J}{N_J} \equiv \bar{\sigma}_{J+1,J}\Delta t_{J+1} = \frac{1}{1+\alpha_{J+1}\Delta t_{J+1}} \frac{P_{J+1}}{P_J} - 1 \tag{1}$$

where $\bar{\sigma}_{J+1,J}$ is defined here as the *time rate of relative change in purchasing ability* over the $(t_J, t_{J+1}]$ time interval.

Fisher Equation. Equation (1) may be used to derive the well-known Fisher Equation. If the time rate of nominal relative change in our P_J monetary amount is known to be σ_{J+1} over $(t_J, t_{J+1}]$, and is assumed constant over the interval, then by definition of σ we may write

$$P_{J+1} = (1+\sigma_{J+1}\Delta t_{J+1})P_J$$

In many investment instruments, such as moneymarket accounts, bank savings accounts, and bond investments, a value for σ is well known. Substituting into (1) gives

$$\bar{\sigma}_{J+1,J}\Delta t_{J+1} = \frac{1+\sigma_{J+1}\Delta t_{J+1}}{1+\alpha_{J+1}\Delta t_{J+1}} - 1 \tag{2}$$

If $\alpha_{J+1}\Delta t_{J+1}$ is small, we may apply a first-order Taylor series approximation to the denominator, yielding a form of the Fisher Equation:

$$\begin{aligned}
\bar{\sigma}_{J+1,J}\Delta t_{J+1} &= (1+\sigma_{J+1}\Delta t_{J+1})\,(1+\alpha_{J+1}\Delta t_{J+1})^{-1} - 1 \\
&\approx (1+\sigma_{J+1}\Delta t_{J+1})\,(1-\alpha_{J+1}\Delta t_{J+1}) - 1 \\
&\approx (\sigma_{J+1}-\alpha_{J+1})\Delta t_{J+1}
\end{aligned}$$

Real price. The real relative change in our purchasing ability (Eq. (1)) over the $(t_J, t_{J+1}]$ time interval admits the notion of a *real price*, $\bar{P}_{i,j}$, applicable at some time t_i, and relative to pricing at some other time t_j. The real price is a fictitious price. It is merely an adjustment to the nominal price of an item to offset the effect of inflation on pricing. To make the adjustment, we simply stipulate that the relative change in the real price over $(t_J, t_{J+1}]$ matches the relative change in our purchasing ability as per Eq. (1). That is, we stipulate

$$\begin{aligned}
\bar{P}_{J,J} &= P_J \\
\frac{\bar{P}_{J+1,J}-\bar{P}_{J,J}}{\bar{P}_{J,J}} &= \frac{N_{J+1}-N_J}{N_J} = \bar{\sigma}_{J+1,J}\Delta t_{J+1}
\end{aligned}$$

The condition $\bar{P}_{J,J} = P_J$ conveys that at some specified initial time t_J we wish for the real price to match the nominal price. Note that throughout this analysis, a bar over a symbol denotes a real quantity instead of a nominal one. The stipulation implies that

$$\begin{aligned}
\bar{P}_{J,J} &= P_J \\
\bar{P}_{J+1,J} &= (1+\bar{\sigma}_{J+1,J}\Delta t_{J+1})\bar{P}_{J,J} = \left(\frac{1+\sigma_{J+1}\Delta t_{J+1}}{1+\alpha_{J+1}\Delta t_{J+1}}\right)\bar{P}_{J,J} \qquad \text{(using Eq. (2))} \\
&= \frac{P_{J+1}}{1+\alpha_{J+1}\Delta t_{J+1}} \qquad \text{(using Eq. (1))} \\
&= \frac{1}{1+\alpha_{J+1}\Delta t_{J+1}} \frac{P_{J+1}}{P_J} \bar{P}_{J,J}
\end{aligned} \tag{3}$$

Next, consider a subsequent time t_{J+2}. At t_{J+2}, our nominal monetary amount is P_{J+2}, say. Again, unfortunately for us as holders of P_{J+2}, the supplier has deemed that the N_J products or services are no longer worth P_J, but instead are worth $(1+\alpha_{J+2}\Delta t_{J+2})P_{J+1}$, where α_{J+2} is the inflation rate at t_{J+2}. So to purchase the N_J items from the supplier, we would have to dispense with $(1+\alpha_{J+2}\Delta t_{J+2})(1+\alpha_{J+1}\Delta t_{J+1})P_J$ units of our currency. And stated reciprocally, at t_{J+2} a single unit of our currency would now be able to purchase only $N_J/[(1+\alpha_{J+2}\Delta t_{J+2})(1+\alpha_{J+1}\Delta t_{J+1})P_J]$ items. But since we now have P_{J+2} units of currency, we are in fact able to purchase

$$N_{J+2} = \frac{N_J}{(1+\alpha_{J+1}\Delta t_{J+1})(1+\alpha_{J+2}\Delta t_{J+2})P_J} P_{J+2}$$

such items of product or service.

To obtain a sensible real price at t_{J+2}, we once again stipulate that the relative change of the real price over the $(t_J, t_{J+2}]$ time interval matches the relative change of our purchasing ability over that same interval.

$$\bar{P}_{J,J} = P_J$$

$$\frac{\bar{P}_{J+2,J} - \bar{P}_{J,J}}{\bar{P}_{J,J}} = \frac{N_{J+2} - N_J}{N_J}$$

From this we obtain, using Eq. (3)

$$\begin{aligned} \bar{P}_{J+2,J} &= \frac{P_{J+2}}{(1+\alpha_{J+1}\Delta t_{J+1})(1+\alpha_{J+2}\Delta t_{J+2})} \\ &= \frac{1}{1+\alpha_{J+2}\Delta t_{J+2}} \frac{P_{J+2}}{P_{J+1}} \bar{P}_{J+1,J} \end{aligned} \tag{4}$$

Following similar reasoning, it is easy to show that a sensible real price at yet a subsequent time t_{J+3}, relative to pricing at t_J, is

$$\begin{aligned} \bar{P}_{J+3,J} &= \frac{P_{J+3}}{(1+\alpha_{J+1}\Delta t_{J+1})(1+\alpha_{J+2}\Delta t_{J+2})(1+\alpha_{J+3}\Delta t_{J+3})} \\ &= \frac{1}{1+\alpha_{J+3}\Delta t_{J+3}} \frac{P_{J+3}}{P_{J+2}} \bar{P}_{J+2,J} \end{aligned} \tag{5}$$

By extension of (3), (4) and (5) to the $(t_{J+k-1}, t_{J+k}]$ time interval, a sensible real price at t_{J+k}, relative to pricing at t_J, is

$$\boxed{\begin{aligned} \bar{P}_{J+k,J} &= \frac{P_{J+k}}{\prod_{l=1}^{k}(1+\alpha_{J+l}\Delta t_{J+l})} \\ &= \frac{1}{1+\alpha_{J+k}\Delta t_{J+k}} \frac{P_{J+k}}{P_{J+k-1}} \bar{P}_{J+k-1,J} \\ &\qquad \text{for } k = 1, 2, \ldots, \text{ with } \bar{P}_{J,J} = P_J \end{aligned}} \tag{6}$$

where $\Delta t_{J+l} \equiv t_{J+l} - t_{J+l-1}$. The second form of the specification of $\bar{P}_{J+k,J}$ in (6) is a recursive implicit one in that the real price at time t_{J+k} depends on the real price at time t_{J+k-1}, namely, $\bar{P}_{J+k-1,J}$. So if $\bar{P}_{J+k-1,J}$ is known, then the second form offers a computational advantage over the first form because the second form requires fewer floating point calculations to obtain $\bar{P}_{J+k,J}$.

Backward real pricing

Alternatively, suppose we wish to calculate historical real prices relative to a current or future price. Then we implicitly agree that the real price must match the nominal price at some time t_J, for $J > 0$. Nominal prices at times earlier than t_J will then be adjusted for inflation to admit real prices relative to that at t_J.

So we begin as before by setting $\bar{P}_{J,J} = P_J$. And following similar reasoning to that in Section , we arrive at

$$\begin{aligned} \bar{P}_{J-k,J} &= \prod_{l=0}^{k-1} (1 + \alpha_{J-l}\Delta t_{J-l})\, P_{J-k} \\ &= (1 + \alpha_{J-k+1}\Delta t_{J-k+1})\, \frac{P_{J-k}}{P_{J-k+1}}\, \bar{P}_{J-k+1,J} \\ &\qquad \text{for } k = 1, 2, \ldots, J, \text{ with } \bar{P}_{J,J} = P_J \end{aligned} \tag{7}$$

US nominal, real and real per-capita GDP

The US inflation rate over the last 155 years is shown in Figure 1. The US nominal Gross Domestic Product and a US real Gross Domestic Product over the last 95 years is shown together in Figure 2. The real history curve was computed relative to a recent year using equation (7) and the data in Figure 1. The US population over the last 95 years is shown in Figure 3. Finally, equation (7) and the data in Figures 1, 2, 3 were used to compute the US real per-capita Gross Domestic Product over the same time period. I did not expect the history to be approximately linear over the 95-year span.

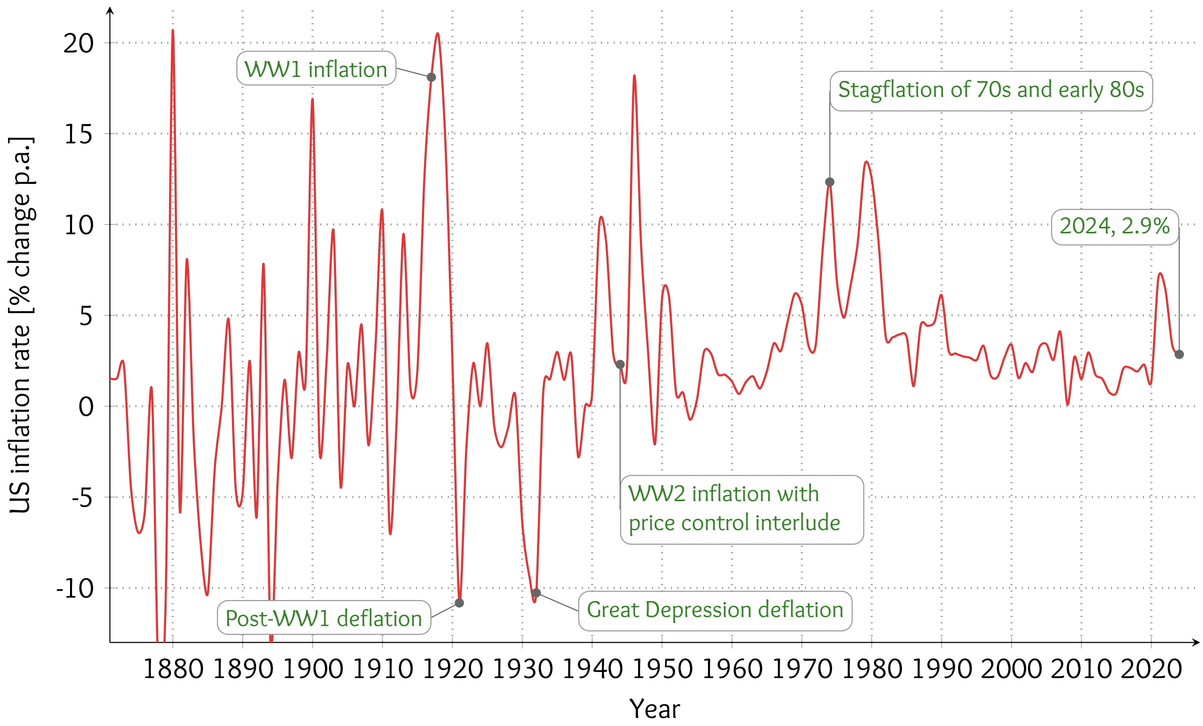

Figure 1: History of the US inflation rate, α_K, based on the Consumer Price Index for all urban consumers. The time averaged inflation rate is 2.3% p.a. (Data sources: Federal Reserve Bank of St Louis,[1] Yale University.[2])

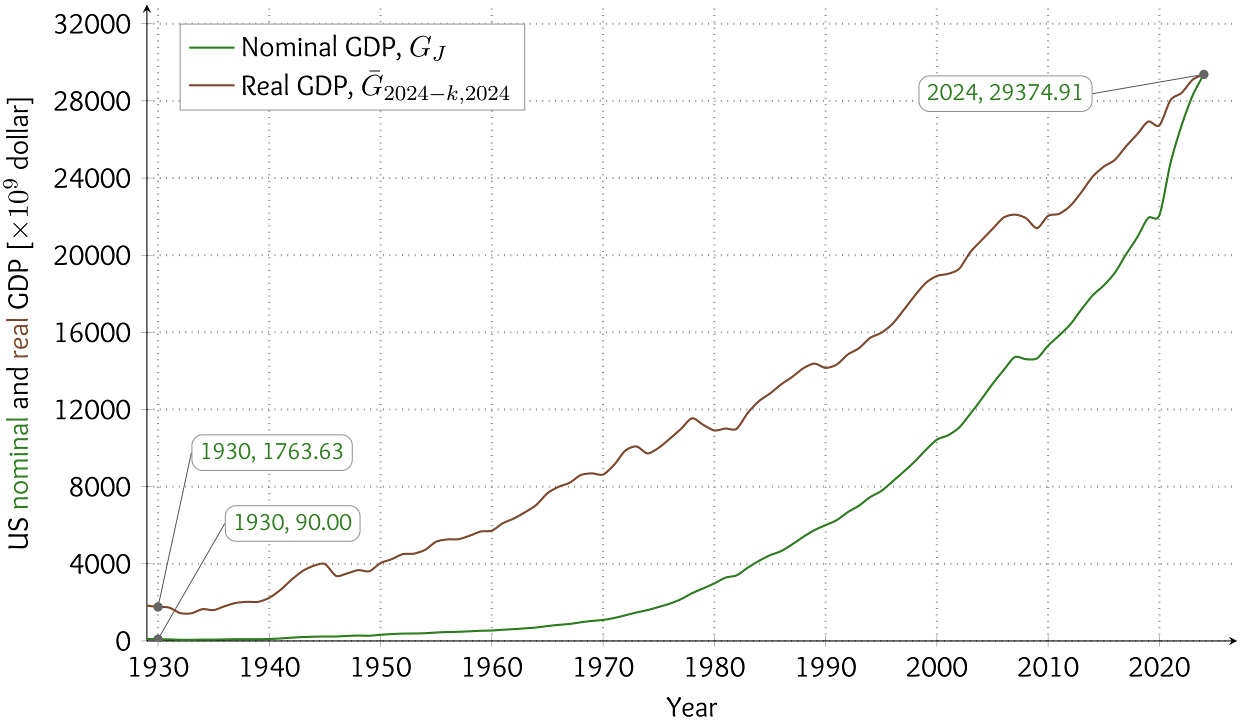

Figure 2: History of the US nominal (i.e., inflation unadjusted) Gross Domestic Product, G_J, and the US real (i.e., inflation adjusted) Gross Domestic Product, $\bar{G}_{2024-k,2024}$, relative to the value in 2024. This real GDP history was computed using equation (7). (Data sources: Federal Reserve Bank of St Louis,[1] `www.multpl.com`,[3] Yale University.[2])

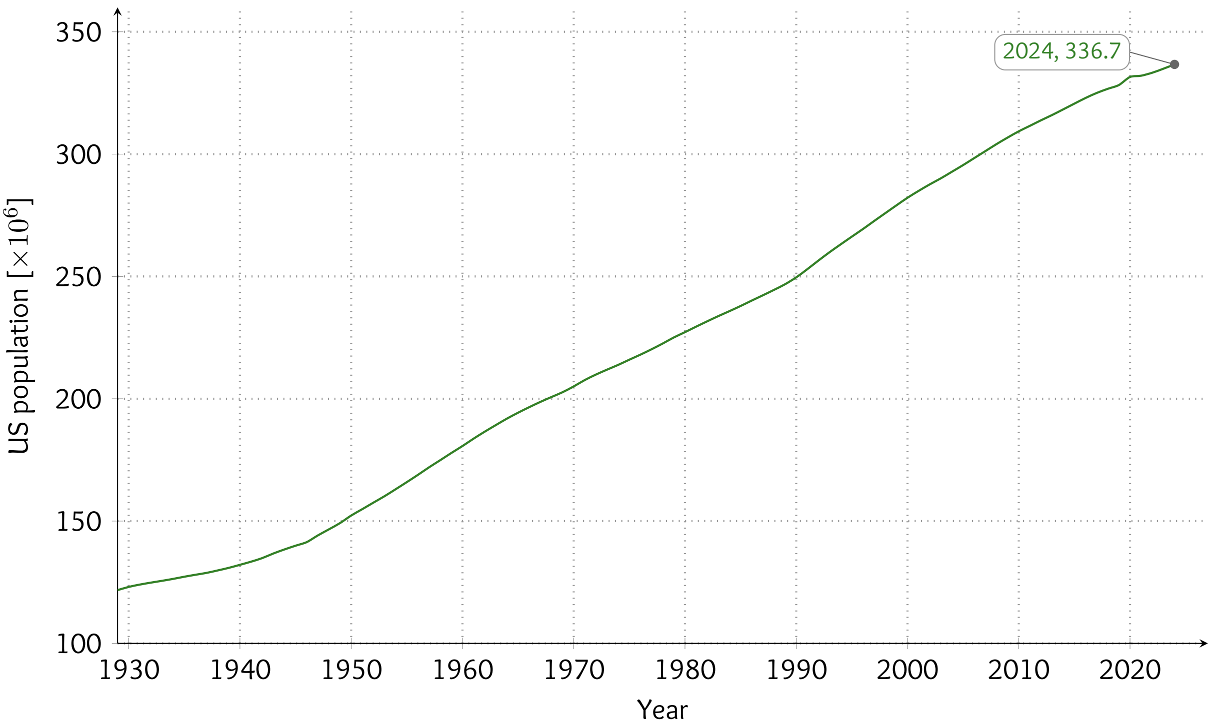

Figure 3: History of the US population. (Data sources: `www.census-charts.com`,[4] `www.multpl.com`.[5])

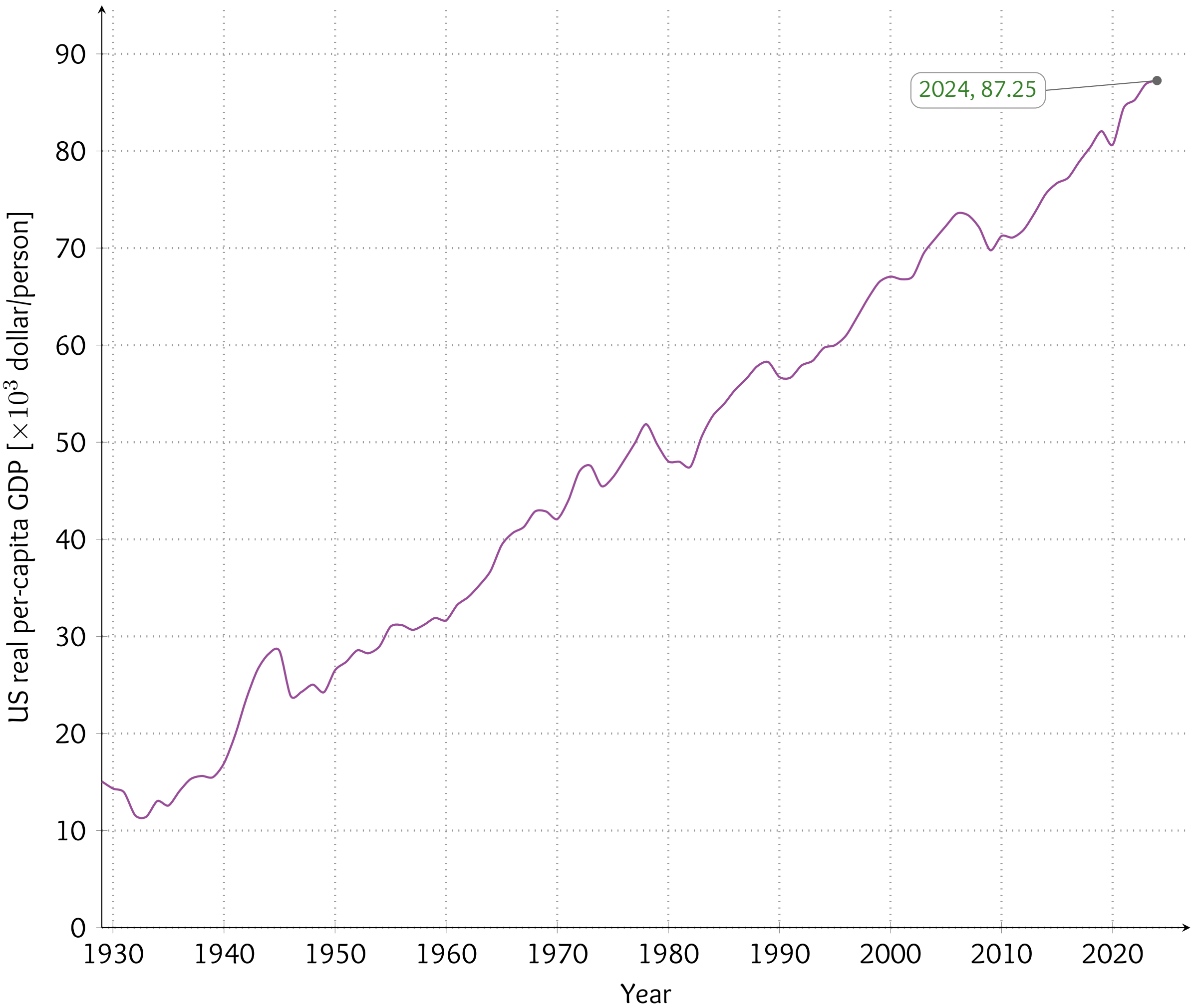

Figure 4: History of the US real (i.e., inflation adjusted) per-capita Gross Domestic Product, $\bar{g}_{2024-k,2024}$, relative to the value in 2024. (Data sources: Federal Reserve Bank of St Louis,[1] `www.multpl.com`,[5, 3] Yale University.[2])

References

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