On Financial Security and Retirement

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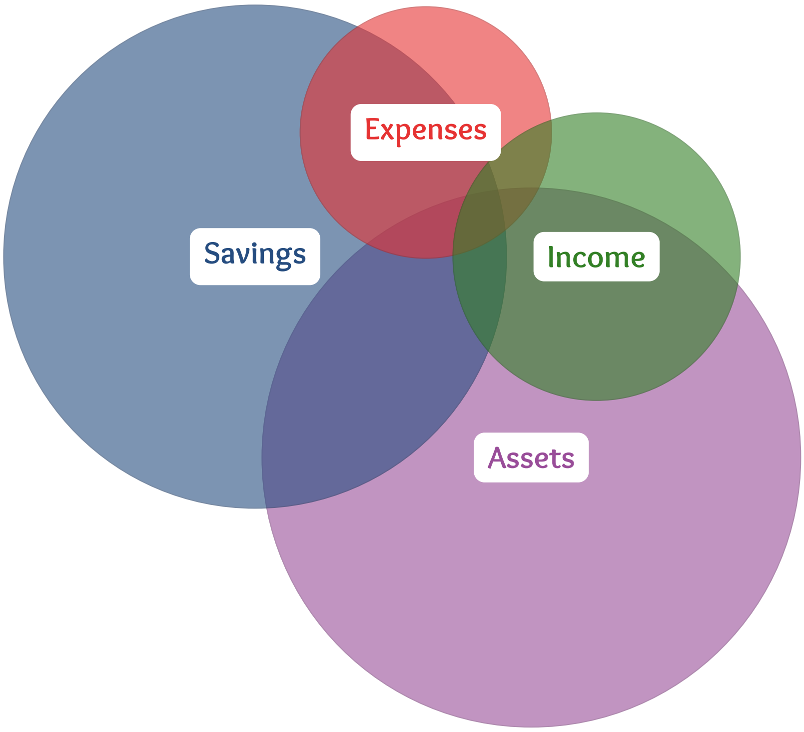

Abstract

THE FINANCIAL SECURITY and well-being[1] of an individual investor—actually, of any person—are determined in part by the investor's overall net monetary worth. Net monetary worth in turn is determined by the interplay of financial variables such as *income from labour*, *living expenses*, *savings*, the *possession of assets* such as a property or a business, *taxes*, and *borrowings*. Net monetary worth is therefore an important focal point in this study. By quantifying it, by tracking its change over time, and by imposing one or more constraints on it, this study demonstrates how an investor's financial security and well-being may be measured numerically. Along the way, important measures of present and/or future financial security are introduced. The numerical analysis presented here was applied in three case studies: 1. The impact on net monetary worth of retaining versus selling an income-generating asset. 2. The impact of a rising expense rate on an investor's present and future financial security. 3. How present financial circumstances affect future financial security before and during retirement.

[1] I declare this to be my own work, entirely. In particular, no AI was used in any research, analysis, synthesis, writing, nor typesetting of this work. In short, AI was not recruited at any time in this work. Errors and inaccuracies are therefore proudly my own.

Contents

1 Introduction 2

2 Net monetary worth 2

2.1 Monetary balance equation 2

2.2 Prudent saver 4

2.3 Financial security before retirement 5

2.4 Financial security after retirement 7

3 A time-discontinuous approximation to net monetary worth 9

4 Case studies 12

4.1 Asset or savings 12

4.2 Rising expense rates 14

4.3 Retirement 17

5 Acknowledgments 23

1 Introduction

How financially secure are we? Are we saving enough for retirement? When will we be able to retire? And if we sell or buy an asset such as a property or a business, how will our overall financial circumstance be altered? Will it leave us more financially secure? Or less? These are all important financial questions. In this study, I offer some of my own answers.

Some of the variables that play important roles in determining financial circumstance are: *income from labour*, *living expenses*, *savings*, the *possession of assets* such as a property or a business, *taxes*, and *borrowings*. These variables each contribute to a monetary balance wherein an investor's *net monetary worth* at any given time is the sum of savings and assets, less liabilities.

So, given a net monetary worth at a certain time, we wish to be able to predict a net monetary worth at future times. Doing so will help us learn the impact of singular events in our financial trajectory. Such events could include: 1. the onset of retirement, characterised by an abrupt change in income; 2. an abrupt change in living expenses; 3. the purchase or sale of an asset; or 4. the use of an existing asset to supplement income, such as a rental income. By carefully quantifying net monetary worth, by tracking its change over time, and by imposing one or more constraints on it, these events may be simulated. I believe these simulations may be carried out with sufficient accuracy as to yield valuable information—information which may be used to alter a person's life trajectory in a positive way.

So let's begin...

2 Net monetary worth

2.1 Monetary balance equation

We begin with a *monetary balance equation*. Let the aggregate value of *savings* be denoted by S, and the aggregate value of *assets* by A. An investor's *net monetary worth* at some time t, denoted by W, is then, simply

$$W(t) = S(t) + A(t) \tag{1}$$

In this analysis, assets, A, refer to the aggregate of any physical assets, such as properties or businesses. Savings, S, typically refers to the aggregate of investment instruments that are deemed more tradeable than a non-fungible asset. Fungible assets are assets that can readily be exchanged or substituted for another asset of equivalent value. Funds placed in an interest-bearing bank account, equities, and bonds are all considered here to be savings.

But there is more to savings here. In this analysis, liabilities are to be considered negative savings too. It is obvious that a positive balance in a bank account is a form of savings. But so too is a negative balance. In the former case, the investor lends to the bank for which the bank compensates the investor with interest. While in the latter case, the bank lends to the investor who then compensates the bank with interest, ostensibly at a different rate. A home loan is also a savings instrument, but with a negative balance. Importantly, the interest rate associated with the home loan is not to be identified with the rate of asset appreciation of the property itself. So, with the understanding that the aggregate of investment savings includes liabilities, there is no need to explicitly account for liabilities in the monetary balance in (1).

The net monetary worth, W, in (1) is an all important quantity in assessing financial circumstance for an individual. Keeping track of how W varies in time requires keeping track of how its components S and A vary. And to do that, we must know the time rates of change of these components. Let σ be the *instantaneous time rate of relative increase in savings*. This might be an interest rate associated with an interest-bearing bank account, or it might be the time rate of relative increase in value of an equity portfolio (with dividends reinvested). The usefulness of σ is that it allows us to express the time rate of change of savings in proportion with the magnitude of savings itself:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \sigma(t)S(t) \tag{2}$$

Similarly, let α be the *instantaneous time rate of relative increase in the value of assets*. This might simply be the appreciation rate of a property. The time rate of change in the value of assets may then be expressed in proportion with the value of assets themselves:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \alpha(t)A(t) \tag{3}$$

Thus far, we have neglected the receipt of income, I, and the payment of expenses, E, as variables that may contribute to W. In practice, income is normally received discontinuously in the form of a salary or when customers settle invoices (See Section 3 on page 9 for the discontinuous approximate analysis). But in keeping with this continuous analysis here, we assume that income is received continuously at a time rate of $\dot{I}$. It is reasonable to assume that the time rate of change of $\dot{I}$ is proportional to $\dot{I}$ itself. For, suppose that income is received at a rate \$100/m (\$100 per month), say. A while later, we might expect income to be received at \$103/m. And still later, at \$108/m. But if, instead, income is presently being received at a rate \$1000/m, then a while later we would expect it to be \$1030/m, and not \$1003/m. The time rate of change of $\dot{I}$ may therefore be expressed in proportion with the rate of receipt of income itself:

$$\frac{\mathrm{d}\dot{I}}{\mathrm{d}t} = \eta(t)\dot{I}(t) \tag{4}$$

Here, η captures this proportionality. It is the *instantaneous time rate of relative increase in the rate of receipt of income*. If we know η over some time interval of interest $[t_0, t]$, then (4) offers a formula for $\dot{I}$ at the end of that same time interval. Since (4) is a simple linear homogenous differential equation, it may easily be solved by integration:

$$\int_{\dot{I}(t_0)}^{\dot{I}(t)} \frac{\mathrm{d}\dot{I}}{\dot{I}} = \int_{t_0}^{t} \eta(\tau)\mathrm{d}\tau$$

$$\ln\Big(\frac{\dot{I}(t)}{\dot{I}(t_0)}\Big) = \int_{t_0}^{t} \eta(\tau)\mathrm{d}\tau$$

which gives

$$\dot{I}(t) = \dot{I}(t_0)e^{\int_{t_0}^{t} \eta(\tau)\mathrm{d}\tau} \tag{5}$$

This may be integrated again to give a formula for the receipt of total income, I, over the $[t_0, t]$ time interval as

$$I(t) = \int_{t_0}^{t} \dot{I}(\tau)\mathrm{d}\tau = I(t_0) + \dot{I}(t_0) \int_{t_0}^{t} e^{\int_{t_0}^{\tau} \eta(\bar{\tau})\mathrm{d}\bar{\tau}} \mathrm{d}\tau \tag{6}$$

To keep the ensuing analysis simple and tractable, we now make a bold assumption, namely, that η is a constant in time. That is, $\eta(t) = \eta$ for all t. This is not an unreasonable assumption, however, because if we place ourselves at $t = t_0$, and project into the future, then we do not know what future values η would take. Under this assumption, (5) reduces to

$$\dot{I}(t) = \dot{I}(t_0)e^{\eta(t-t_0)} \tag{7}$$

And (6) reduces to

$$I(t) = I(t_0) + \frac{\dot{I}(t_0)}{\eta}\left(e^{\eta(t-t_0)} - 1\right) \tag{8}$$

Similar to income, in keeping with this continuous analysis, assume that expenses are incurred continuously over time at a rate $\dot{E}$. If a certain good or service presently requires us to spend at a rate \$100/m (\$100 per month), say, then a while later, we might expect to spend at the increased rate \$103/m for the same good or service. But if another good or service presently requires us to spend at a rate of \$1000/m, say, then a while later, we would expect to spend at the rate \$1030/m, and not \$1003/m. The time rate of change of $\dot{E}$ may therefore be expressed in proportion with the rate of expenditure of itself:

$$\frac{\mathrm{d}\dot{E}}{\mathrm{d}t} = \varepsilon(t)\dot{E}(t) \tag{9}$$

for some parameter ε. ε is an instantaneous notion of inflation because it pertains to increases in living expenses due solely to the increase in the prices of goods and services, and not by any change in expenditure patterns. Following (4), (9) offers the solution

$$\dot{E}(t) = \dot{E}(t_0)e^{\int_{t_0}^{t} \varepsilon(\tau)\mathrm{d}\tau}$$

Once again, making a further assumption that $\varepsilon(t) = \varepsilon$ for any t gives

$$\dot{E}(t) = \dot{E}(t_0)e^{\varepsilon(t-t_0)} \tag{10}$$

Integrating again gives the total expenditure over the $[t_0, t]$ time interval as

$$E(t) = E(t_0) + \frac{\dot{E}(t_0)}{\varepsilon}\left(e^{\varepsilon(t-t_0)} - 1\right)$$

The monetary balance in (1) makes no mention of income and expenses, and yet they must influence the balance in some way. To introduce their influence on the monetary balance, we make an arguably reasonable assumption about investor behaviour. We assume that the investor is a prudent saver.

2.2 Prudent saver

Income from labour typically accrues in a zero- or low-interest transactional bank account, and so is considered to be separate from savings. Living expenses are assumed to be incurred from the same transactional bank account. It is now assumed that the investor is a *prudent saver*. A prudent saver will attempt to maximise net monetary worth by injecting the difference element $(\mathrm{d}I - \mathrm{d}E)$ at time t into the savings element, $\mathrm{d}S$. This might be as simple as transferring the difference $(\mathrm{d}I - \mathrm{d}E)$ from a transactional bank account into a moneymarket savings account. Under this assumption, the revised element of savings becomes

$$\mathrm{d}S = \sigma S\mathrm{d}t + \mathrm{d}I - \mathrm{d}E = \sigma S\mathrm{d}t + \dot{I}\mathrm{d}t - \dot{E}\mathrm{d}t$$

so that using (7) and (10), a revised form of (2) is

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \sigma(t)S(t) + \dot{I}(t_0)e^{\eta(t-t_0)} - \dot{E}(t_0)e^{\varepsilon(t-t_0)}$$

This can be solved for $S(t)$ by exploiting $e^{-\sigma t}$ as a so-called integrating factor

$$\begin{aligned}
&\frac{\mathrm{d}S}{\mathrm{d}t} - \sigma S = \dot{I}(t_0)e^{\eta(t-t_0)} - \dot{E}(t_0)e^{\varepsilon(t-t_0)} \\
&e^{-\sigma t}\frac{\mathrm{d}S}{\mathrm{d}t} - \sigma e^{-\sigma t}S = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\sigma t}S\right) = e^{-\sigma t}\left(\dot{I}(t_0)e^{\eta(t-t_0)} - \dot{E}(t_0)e^{\varepsilon(t-t_0)}\right) \\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\sigma t}S\right) = \dot{I}(t_0)e^{(\eta-\sigma)t-\eta t_0} - \dot{E}(t_0)e^{(\varepsilon-\sigma)t-\varepsilon t_0}
\end{aligned}$$

Integrating gives

$$\begin{aligned}
e^{-\sigma t}S(t) - e^{-\sigma t_0}S(t_0) &= \frac{\dot{I}(t_0)}{\eta-\sigma}\left(e^{(\eta-\sigma)t-\eta t_0} - e^{(\eta-\sigma)t_0-\eta t_0}\right) - \frac{\dot{E}(t_0)}{\varepsilon-\sigma}\left(e^{(\varepsilon-\sigma)t-\varepsilon t_0} - e^{(\varepsilon-\sigma)t_0-\varepsilon t_0}\right) \\
&= \frac{\dot{I}(t_0)}{\eta-\sigma}\left(e^{(\eta-\sigma)t-\eta t_0} - e^{-\sigma t_0}\right) - \frac{\dot{E}(t_0)}{\varepsilon-\sigma}\left(e^{(\varepsilon-\sigma)t-\varepsilon t_0} - e^{-\sigma t_0}\right)
\end{aligned}$$

from which we obtain

$$S(t) = S(t_0)e^{\sigma(t-t_0)} + \frac{\dot{I}(t_0)}{\eta-\sigma}\left(e^{\eta(t-t_0)} - e^{\sigma(t-t_0)}\right) - \frac{\dot{E}(t_0)}{\varepsilon-\sigma}\left(e^{\varepsilon(t-t_0)} - e^{\sigma(t-t_0)}\right) \tag{11}$$

To obtain a tractable expression for the value of assets at any time t, we make yet another assumption, namely, that $\alpha(t) = \alpha$ for any t. Equation (3) then results in

$$A(t) = A(t_0)e^{\alpha(t-t_0)} \tag{12}$$

Substituting (11) and (12) into (1), we arrive at an expression for an investor's future net monetary worth as

$$\boxed{\begin{aligned}
W(t) = S(t_0)e^{\sigma(t-t_0)} &+ A(t_0)e^{\alpha(t-t_0)} \\
&+ \frac{\dot{I}(t_0)}{\eta-\sigma}\left(e^{\eta(t-t_0)} - e^{\sigma(t-t_0)}\right) \\
&- \frac{\dot{E}(t_0)}{\varepsilon-\sigma}\left(e^{\varepsilon(t-t_0)} - e^{\sigma(t-t_0)}\right) \qquad \text{for } t \geqslant t_0
\end{aligned}} \tag{13}$$

I think this is an important result. It exposes the influences of the present values of savings $S(t_0)$, of non-fungible assets $A(t_0)$, of income receipt rate $\dot{I}(t_0)$, and of expenses rate $\dot{E}(t_0)$ to a net monetary worth W at some future time t.

2.3 Financial security before retirement

The importance of (13) notwithstanding, people will likely not agree on what it means to be financially secure. Indeed, older, younger, richer and poorer people might all attach different meanings. Therefore, many possible candidate criteria should exist for quantifying financial security.

One obvious candidate criterion is for an individual's present income from labour, $I(t_0)$, to exceed present expenses, $E(t_0)$. This criterion is obviously fine for stipulating a short-term financial security. But the criterion is inadequate because it precludes any consideration of a future retirement event. At retirement, income from labour usually vanishes because labour ceases. So after retirement, other income streams need to be tapped in order to cover expenses. These income streams could come from a pension, from savings, from assets, or from a combination thereof.

Another candidate criterion of financial security is the condition that a person's net monetary worth reduces to zero only at some time after a person dies. Of course, application of the criterion requires being able to reliably predict a person's date of death. It also requires predicting values for future economic parameters, such as the rate of inflation, the rate of interest accrual, and the rate of asset appreciation. This dependence on possibly far-futured variables limits the reliability of the condition.

I hereby offer an alternative criterion for financial security, albeit a somewhat stringent one, namely, the condition that a person's *net monetary worth grows at a rate greater than or equal to the rate of inflation*. Satisfying this condition should compensate for the inevitable and insidious erosion of the value of net monetary worth over time as goods and services seem to become ever more expensive. It is also a condition that can be applied at any moment in a person's financial trajectory. So, to demand that net monetary worth W at some time t grows at a rate greater than or equal to the inflation rate at that same time means to demand that

$$W(t + \mathrm{d}t) - W(t) \geqslant \varepsilon(t)W(t)\mathrm{d}t$$

That is, to demand that

$$\frac{\mathrm{d}W(t)}{\mathrm{d}t} \geqslant \varepsilon(t)W(t)$$

And using (13), it means to demand that

$$\begin{aligned}\frac{\mathrm{d}W(t)}{\mathrm{d}t} &= \sigma S(t_0)e^{\sigma(t-t_0)} + \alpha A(t_0)e^{\alpha(t-t_0)} + \frac{\dot{I}(t_0)}{\eta - \sigma}\left(\eta e^{\eta(t-t_0)} - \sigma e^{\sigma(t-t_0)}\right) \\ &\qquad - \frac{\dot{E}(t_0)}{\varepsilon - \sigma}\left(\varepsilon e^{\varepsilon(t-t_0)} - \sigma e^{\sigma(t-t_0)}\right) \\ &\geqslant \varepsilon\Big[S(t_0)e^{\sigma(t-t_0)} + A(t_0)e^{\alpha(t-t_0)} + \frac{\dot{I}(t_0)}{\eta - \sigma}\left(e^{\eta(t-t_0)} - e^{\sigma(t-t_0)}\right) \\ &\qquad - \frac{\dot{E}(t_0)}{\varepsilon - \sigma}\left(e^{\varepsilon(t-t_0)} - e^{\sigma(t-t_0)}\right)\Big]\end{aligned}$$

Therefore, the *financial security criterion*, $C_{\text{sec}}(t; t_0)$, that net monetary worth at some future time t grows at a rate greater than or equal to the inflation rate is

$$\boxed{\begin{aligned}C_{\text{sec}}(t; t_0) \equiv (\sigma - \varepsilon)S(t_0)e^{\sigma(t-t_0)} &+ (\alpha - \varepsilon)A(t_0)e^{\alpha(t-t_0)} \\ &+ \frac{\dot{I}(t_0)}{\eta - \sigma}\left((\eta - \varepsilon)e^{\eta(t-t_0)} - (\sigma - \varepsilon)e^{\sigma(t-t_0)}\right) \\ &- \dot{E}(t_0)e^{\sigma(t-t_0)} \\ \geqslant\ 0 \qquad\qquad & \text{for } t \geqslant t_0\end{aligned}} \tag{14}$$

Satisfying this constraining relationship between the present values of $S(t_0)$, $A(t_0)$, $\dot{I}(t_0)$ and $\dot{E}(t_0)$ will maximise the likelihood of long-term financial security at some future specified time t. The constraining relationship captures the intuitive idea that savings, asset values and income from labour should be maximised, while expenses should be minimised if the investor wishes to increase the odds of long-term financial security. Increasing present savings, $S(t_0)$, for example, will work to increase $C_{\text{sec}}(t; t_0)$ for some future time t, provided that $\sigma - \varepsilon$ is positive. So this difference between the savings rate σ and the inflation rate ε also influence financial security. Being the coefficient of the "$S(t_0)$" term in (14), as $\sigma - \varepsilon$ increases, $C_{\text{sec}}(t; t_0)$ increases. Similarly for the difference between the asset appreciation rate α and the inflation rate. Note that if we demand financial security now, i.e., at t_0, then the criterion reduces to

$$C_{\text{sec}}(t_0; t_0) = (\sigma - \varepsilon)S(t_0) + (\alpha - \varepsilon)A(t_0) + \dot{I}(t_0) - \dot{E}(t_0)\ \geqslant\ 0$$

The usefulness of $C_{\text{sec}}(t; t_0)$ is that it combines all the financial variables into a single encompassing relation. It does not stipulate in particular what the investor's present savings, $S(t_0)$, nor present assets, $A(t_0)$, should be. But suppose we *are* in fact interested in linking present savings to future financial security. Then we may use (14) to impose a condition on $S(t_0)$ as:

$$\boxed{\begin{aligned}S(t_0)\ \geqslant\ S_{\text{sec}}(t; t_0) \equiv \frac{1}{\sigma - \varepsilon}\Big[&\dot{E}(t_0) - (\alpha - \varepsilon)A(t_0)e^{(\alpha-\sigma)(t-t_0)} \\ &- \frac{\dot{I}(t_0)}{\eta - \sigma}\left((\eta - \varepsilon)e^{(\eta-\sigma)(t-t_0)} - \sigma + \varepsilon\right)\Big] \qquad \text{for } t \geqslant t_0\end{aligned}} \tag{15}$$

This criterion answers the question, "What must our present savings be in order for us to enjoy financial security at some future specified time?" And its answer is, simply, $S(t_0) \geqslant S_{\text{sec}}(t;t_0)$. That is, $S_{\text{sec}}(t;t_0)$ is a prescribed lower bound on $S(t_0)$.

Why use the notation "$S_{\text{sec}}(t;t_0)$" here instead of simply "$S_{\text{sec}}(t_0)$"? The first ordinate, "t", denotes a functional dependence of the criterion on that future time t at which we wish to obtain financial security. The second ordinate, "t_0", denotes that S_{sec} is a constraint on the *present* savings value, $S(t_0)$, not on a future one. But if we are in fact only interested in financial security at the present time, t_0, then the criterion reduces to

$$S(t_0) \ \geqslant \ S_{\text{sec}}(t_0;t_0) = \frac{1}{\sigma-\varepsilon}\Big[\dot{E}(t_0) - \dot{I}(t_0) - (\alpha-\varepsilon)A(t_0)\Big]$$

Finally, we ask, "What must our present rate of expenses be in order for us to enjoy financial security at some future specified time?" Then we may use (14) to impose a condition on $E(t_0)$ as:

$$\boxed{\begin{aligned}\dot{E}(t_0) \ \leqslant \ \dot{E}_{\text{sec}}(t;t_0) \equiv (\sigma-\varepsilon)S(t_0) + (\alpha-\varepsilon)A(t_0)e^{(\alpha-\sigma)(t-t_0)} \\ + \frac{\dot{I}(t_0)}{\eta-\sigma}\big((\eta-\varepsilon)e^{(\eta-\sigma)(t-t_0)} - \sigma + \varepsilon\big) \qquad \text{for } t \geqslant t_0\end{aligned}} \tag{16}$$

2.4 Financial security after retirement

Many people are lucky enough to be able to work for many years in their adult life. Whilst working, they derive an income stream from their labours. Of those people, some dutifully contribute towards some or other pension fund, hoping that when they wish to retire, the pension fund will trigger, replacing part or all of the labour-based income stream which vanished at the moment of retirement.

Conversely, other people might prefer *not* to contribute towards a pension fund, but instead, to contribute to their own investment portfolio. While this strategy might be unusual and even brave, I think it is not unreasonable. For two reasons. Firstly, pension funds typically invest members' contributions into a range of investment asset classes, such as cash, bonds, equities, properties, and listed property. Nowadays, this range of asset classes is readily accessible by ordinary people, so that ordinary people may invest into these same asset classes without the pension fund acting as an intermediary. Secondly, pension fund managers and/or administrators charge ongoing fees, and these fees accumulate, sometimes insidiously. Herein, I consider both preferences, with the latter preference simply being a special case of the former.

Define a future time of retirement to be that time t_R at which income from labour vanishes and is abruptly replaced with a pension income. Pension income may differ in magnitude from labour-based income. To account for the abovementioned latter preference, the pension income can be set to nil. Given the abrupt change in the investor's income stream at retirement, modelling its impact requires that the investor's timeline be partitioned into two contiguous time intervals, $[t_0, t_R)$ and $[t_R, t]$. The first interval spans the investor's working life, and the second spans the investor's retired life.

From (13), at the end of the first interval and just before the onset of retirement, the investor's net monetary worth is projected to be

$$\begin{aligned}W(t_R^-) = S(t_0)e^{\sigma(t_R-t_0)} + A(t_0)e^{\alpha(t_R-t_0)} \\ + \frac{\dot{I}(t_0)}{\eta-\sigma}\big(e^{\eta(t_R-t_0)} - e^{\sigma(t_R-t_0)}\big) \\ - \frac{\dot{E}(t_0)}{\varepsilon-\sigma}\big(e^{\varepsilon(t_R-t_0)} - e^{\sigma(t_R-t_0)}\big)\end{aligned} \tag{17}$$

Recall from (7) that in the first interval, the investor receives an income from labour at a time rate $\dot{I}(t)$:

$$\dot{I}(t) = \dot{I}(t_0)e^{\eta(t-t_0)} \qquad \text{for } t_0 \leqslant t < t_R$$

Similarly, in the second interval, the investor receives a retirement income at a time rate $\dot{I}_R(t)$, say:

$$\dot{I}_R(t) = \dot{I}_R(t_R)e^{\eta(t-t_R)} \qquad \text{for } t \geqslant t_R$$

and where in general, $\dot{I}_{\mathrm{R}}(t_{\mathrm{R}}) \neq \dot{I}(t_{\mathrm{R}})$ but may be nil. Now, applying two simple changes of variable $t_0 \to t_{\mathrm{R}}$ and $\dot{I} \to \dot{I}_{\mathrm{R}}$ in (13), the investor's net monetary worth at any time $t \geqslant t_{\mathrm{R}}$ must be

$$\begin{aligned} W_{\mathrm{R}}(t) = S(t_{\mathrm{R}})e^{\sigma(t-t_{\mathrm{R}})} &+ A(t_{\mathrm{R}})e^{\alpha(t-t_{\mathrm{R}})} \\ &+ \frac{\dot{I}_{\mathrm{R}}(t_{\mathrm{R}})}{\eta-\sigma}\big(e^{\eta(t-t_{\mathrm{R}})} - e^{\sigma(t-t_{\mathrm{R}})}\big) \\ &- \frac{\dot{E}(t_{\mathrm{R}})}{\varepsilon-\sigma}\big(e^{\varepsilon(t-t_{\mathrm{R}})} - e^{\sigma(t-t_{\mathrm{R}})}\big) \qquad \text{for } t \geqslant t_{\mathrm{R}} \end{aligned}$$

This expression for $W_{\mathrm{R}}(t)$ is all well and good, but it obtains from quantities prevailing at time t_{R}, not t_0. At the onset of retirement, the investor will receive a pension income as a fraction of his/her labour-based income. That is

$$\dot{I}_{\mathrm{R}}(t_{\mathrm{R}}) = \lambda \dot{I}(t_{\mathrm{R}}) \qquad \text{for } 0 \leqslant \lambda \leqslant 1 \tag{18}$$

Of course, to simulate the case of the investor not receiving any pension income stream, we simply set $\lambda = 0$. Substituting (18) together with (10), (11) and (12) at $t = t_{\mathrm{R}}$,

$$\begin{aligned} W_{\mathrm{R}}(t) = &\Big[S(t_0)e^{\sigma(t_{\mathrm{R}}-t_0)} + \frac{\dot{I}(t_0)}{\eta-\sigma}\big(e^{\eta(t_{\mathrm{R}}-t_0)} - e^{\sigma(t_{\mathrm{R}}-t_0)}\big) - \frac{\dot{E}(t_0)}{\varepsilon-\sigma}\big(e^{\varepsilon(t_{\mathrm{R}}-t_0)} - e^{\sigma(t_{\mathrm{R}}-t_0)}\big)\Big]e^{\sigma(t-t_{\mathrm{R}})} \\ &+ \Big[A(t_0)e^{\alpha(t_{\mathrm{R}}-t_0)}\Big]e^{\alpha(t-t_{\mathrm{R}})} \\ &+ \frac{1}{\eta-\sigma}\Big[\lambda\dot{I}(t_0)e^{\eta(t_{\mathrm{R}}-t_0)}\Big]\big(e^{\eta(t-t_{\mathrm{R}})} - e^{\sigma(t-t_{\mathrm{R}})}\big) \\ &- \frac{1}{\varepsilon-\sigma}\Big[\dot{E}(t_0)e^{\varepsilon(t_{\mathrm{R}}-t_0)}\Big]\big(e^{\varepsilon(t-t_{\mathrm{R}})} - e^{\sigma(t-t_{\mathrm{R}})}\big) \qquad \text{for } t \geqslant t_{\mathrm{R}} \end{aligned}$$

which gives

$$\boxed{\begin{aligned} W_{\mathrm{R}}(t) = S(t_0)e^{\sigma(t-t_0)} &+ A(t_0)e^{\alpha(t-t_0)} \\ &+ \frac{\dot{I}(t_0)}{\eta-\sigma}\big((1-\lambda)e^{\eta(t_{\mathrm{R}}-t_0)+\sigma(t-t_{\mathrm{R}})} + \lambda e^{\eta(t-t_0)} - e^{\sigma(t-t_0)}\big) \\ &- \frac{\dot{E}(t_0)}{\varepsilon-\sigma}\big(e^{\varepsilon(t-t_0)} - e^{\sigma(t-t_0)}\big) \qquad \text{for } t \geqslant t_{\mathrm{R}} \end{aligned}} \tag{19}$$

To confirm, it is easy to check that by setting $\lambda = 1$, we obtain $W_{\mathrm{R}}(t) = W(t)$ for all $t \geqslant t_{\mathrm{R}}$, as expected. Also, it is easy to check that for any λ, $W_{\mathrm{R}}(t_{\mathrm{R}}) = W(t_{\mathrm{R}})$ from (17), as expected.

We are interested in obtaining a condition for financial security *during* retirement. Because the investor's timeline has now been partitioned into two contiguous time intervals, we must now demand as before that net monetary worth in the second time interval grows at a rate greater than or equal to the inflation rate. That is, we must now demand that

$$\frac{\mathrm{d}W_{\mathrm{R}}(t)}{\mathrm{d}t} \geqslant \varepsilon W_{\mathrm{R}}(t) \qquad \text{for } t \geqslant t_{\mathrm{R}}$$

Using (19), it means to demand that

$$\begin{aligned} \frac{\mathrm{d}W_{\mathrm{R}}(t)}{\mathrm{d}t} &= \sigma S(t_0)e^{\sigma(t-t_0)} + \alpha A(t_0)e^{\alpha(t-t_0)} \\ &\qquad + \frac{\dot{I}(t_0)}{\eta-\sigma}\big((1-\lambda)\sigma e^{\eta(t_{\mathrm{R}}-t_0)+\sigma(t-t_{\mathrm{R}})} + \lambda\eta e^{\eta(t-t_0)} - \sigma e^{\sigma(t-t_0)}\big) \\ &\qquad - \frac{\dot{E}(t_0)}{\varepsilon-\sigma}\big(\varepsilon e^{\varepsilon(t-t_0)} - \sigma e^{\sigma(t-t_0)}\big) \\ &\geqslant \varepsilon\Big[S(t_0)e^{\sigma(t-t_0)} + A(t_0)e^{\alpha(t-t_0)} \\ &\qquad + \frac{\dot{I}(t_0)}{\eta-\sigma}\big((1-\lambda)e^{\eta(t_{\mathrm{R}}-t_0)+\sigma(t-t_{\mathrm{R}})} + \lambda e^{\eta(t-t_0)} - e^{\sigma(t-t_0)}\big) \\ &\qquad - \frac{\dot{E}(t_0)}{\varepsilon-\sigma}\big(e^{\varepsilon(t-t_0)} - e^{\sigma(t-t_0)}\big)\Big] \qquad \text{for } t \geqslant t_{\mathrm{R}} \end{aligned}$$

Therefore, a revised *financial security criterion*, $C_{\text{Rsec}}(t;t_0)$, which stipulates that net monetary worth *during retirement*, i.e., for any future time $t \geqslant t_{\text{R}}$, must grow at a rate greater than or equal to the inflation rate is

$$\begin{aligned} C_{\text{Rsec}}(t;t_0) \equiv\ & (\sigma-\varepsilon)S(t_0)e^{\sigma(t-t_0)} \\ & +(\alpha-\varepsilon)A(t_0)e^{\alpha(t-t_0)} \\ & +\frac{\dot{I}(t_0)}{\eta-\sigma}\big((1-\lambda)(\sigma-\varepsilon)e^{\eta(t_{\text{R}}-t_0)+\sigma(t-t_{\text{R}})}+\lambda(\eta-\varepsilon)e^{\eta(t-t_0)}-(\sigma-\varepsilon)e^{\sigma(t-t_0)}\big) \\ & -\dot{E}(t_0)e^{\sigma(t-t_0)} \\ \geqslant\ & 0 \qquad\qquad \text{for } t \geqslant t_{\text{R}} \end{aligned} \tag{20}$$

If this constraining relationship between $S(t_0)$, $A(t_0)$, $\dot{I}(t_0)$, $\dot{E}(t_0)$ and λ is satisfied at the present time t_0, then the likelihood of long-term financial security at some future specified time $t \geqslant t_{\text{R}}$, i.e., during retirement, is maximised. In other words, the criterion renders numerically explicit the importance of saving now for retirement later!

It is worth noting that C_{Rsec} in (20) is an extension of C_{sec} in (14). For if we set $\lambda = 1$, then it is straightforward to show that $C_{\text{Rsec}}(t;t_0) = C_{\text{sec}}(t;t_0)$ for all t, which is just the financial security criterion as if there was no transition into retirement. Finally, if we demand financial security at the time t_{R} of retirement, then (20) stipulates that

$$\begin{aligned} C_{\text{Rsec}}(t_{\text{R}};t_0) =\ & (\sigma-\varepsilon)S(t_0)e^{\sigma(t_{\text{R}}-t_0)} \\ & +(\alpha-\varepsilon)A(t_0)e^{\alpha(t_{\text{R}}-t_0)} \\ & +\frac{\dot{I}(t_0)}{\eta-\sigma}\big(((1-\lambda)\sigma+\lambda\eta-\varepsilon)e^{\eta(t_{\text{R}}-t_0)}-(\sigma-\varepsilon)e^{\sigma(t_{\text{R}}-t_0)}\big) \\ & -\dot{E}(t_0)e^{\sigma(t_{\text{R}}-t_0)} \\ \geqslant\ & 0 \end{aligned}$$

This financial security criterion at the time of retirement is ostensibly different from $C_{\text{sec}}(t_0;t_{\text{R}})$. $C_{\text{Rsec}}(t_{\text{R}};t_0)$ correctly accounts for the possibility of an abrupt change in income receipt rate ($\lambda < 1$) at the onset of retirement.

Following (15), suppose we are interested in ensuring that our present savings are sufficient for a financially secure retirement. Then we may easily use (20) to impose a condition on our present savings $S(t_0)$ as

$$\begin{aligned} S(t_0)\ \geqslant S_{\text{Rsec}}(t;t_0) \equiv \frac{1}{\sigma-\varepsilon}\Big[\dot{E}(t_0)-(\alpha-\varepsilon)A(t_0)e^{(\alpha-\sigma)(t-t_0)} \\ -\frac{\dot{I}(t_0)}{\eta-\sigma}\big((1-\lambda)(\sigma-\varepsilon)e^{(\eta-\sigma)(t_{\text{R}}-t_0)}+\lambda(\eta-\varepsilon)e^{(\eta-\sigma)(t-t_0)}-\sigma+\varepsilon\big)\Big] \quad \text{for } t \geqslant t_{\text{R}} \end{aligned} \tag{21}$$

Alternatively, following (16), suppose we are interested in knowing what our present rate of incurred expenses must be to allow for a financially secure future retirement. Then we may easily use (20) to impose a condition on our present rate of expenses $E(t_0)$ as

$$\begin{aligned} \dot{E}(t_0)\ \leqslant \dot{E}_{\text{Rsec}}(t;t_0) \equiv (\sigma-\varepsilon)S(t_0)+(\alpha-\varepsilon)A(t_0)e^{(\alpha-\sigma)(t-t_0)} \\ +\frac{\dot{I}(t_0)}{\eta-\sigma}\big((1-\lambda)(\sigma-\varepsilon)e^{(\eta-\sigma)(t_{\text{R}}-t_0)}+\lambda(\eta-\varepsilon)e^{(\eta-\sigma)(t-t_0)}-\sigma+\varepsilon\big) \quad \text{for } t \geqslant t_{\text{R}} \end{aligned} \tag{22}$$

3 A time-discontinuous approximation to net monetary worth

INCOME FROM LABOUR is usually received at specific moments, i.e., not continuously in time. And in the case of a salary, these discrete moments are equally spaced, typically once per month. How then can the above time-continuous formulation be modified to better accommodate such discreteness in time? In particular, can we use (13) to arrive at a corresponding time-discrete form of (13), (14), (15) and (16)?

We begin with (13), shift the reference instant from t_0 to t_n, and ask what the net monetary worth will be at instant t_{n+1}.

$$\begin{aligned} W(t_{n+1}) = S(t_n)e^{\sigma(t_{n+1}-t_n)} &+ A(t_n)e^{\alpha(t_{n+1}-t_n)} \\ &+ \frac{\dot{I}(t_n)}{\eta-\sigma}\big(e^{\eta(t_{n+1}-t_n)} - e^{\sigma(t_{n+1}-t_n)}\big) \\ &- \frac{\dot{E}(t_n)}{\varepsilon-\sigma}\big(e^{\varepsilon(t_{n+1}-t_n)} - e^{\sigma(t_{n+1}-t_n)}\big) \end{aligned}$$

But since $W(t_n) = S(t_n) + A(t_n)$ (Equation (1)),

$$\begin{aligned} W(t_{n+1}) = W(t_n)e^{\sigma(t_{n+1}-t_n)} &+ A(t_n)\big(e^{\alpha(t_{n+1}-t_n)} - e^{\sigma(t_{n+1}-t_n)}\big) \\ &+ \frac{\dot{I}(t_n)}{\eta-\sigma}\big(e^{\eta(t_{n+1}-t_n)} - e^{\sigma(t_{n+1}-t_n)}\big) \\ &- \frac{\dot{E}(t_n)}{\varepsilon-\sigma}\big(e^{\varepsilon(t_{n+1}-t_n)} - e^{\sigma(t_{n+1}-t_n)}\big) \end{aligned}$$

We now assume that the moments of receipt of income and of payments of expenses are equally spaced. That is, $t_{n+1} - t_n = \Delta t$ for all n. Carrying out Taylor series expansions,

$$W(t_{n+1}) = W(t_n)(1+\sigma\Delta t) + A(t_n)(\alpha-\sigma)\Delta t + \dot{I}(t_n)\Delta t - \dot{E}(t_n)\Delta t + o(\Delta t^2) \quad (23)$$

Considering the $A(t_n)$ term in (23) and using (12) gives

$$A(t_n) = A(t_{n-1})e^{\alpha(t_n-t_{n-1})} = A(t_{n-1})e^{\alpha\Delta t} = A(t_{n-1})(1+\alpha\Delta t) + o(\Delta t^2)$$

This gives the approximate discrete formula for $A(t_n)$

$$A(t_n) \approx A_n = (1+\alpha\Delta t)A_{n-1} = (1+\alpha\Delta t)^2 A_{n-2} = \cdots = (1+\alpha\Delta t)^n A_0$$

Next, consider the $\dot{I}(t_n)$ term in (23). By the change of variable $t_0 \to t_n$ and setting $t = t_{n+1}$ in (8), we have

$$\begin{aligned} \dot{I}(t_n)\Delta t &= \big(I(t_{n+1}) - I(t_n)\big)\frac{\eta}{e^{\eta\Delta t}-1}\Delta t \\ &\approx \big(I(t_{n+1}) - I(t_n)\big)\frac{\eta}{1+\eta\Delta t-1}\Delta t \\ &= I(t_{n+1}) - I(t_n) \\ &\equiv \Delta I_{n+1} \end{aligned}$$

Equation (4) offered a time-continuous prescription of $\dot{I}(t_{n+1})$ as a function of $\dot{I}(t_n)$, with the essential parameter η being a relative time rate quantity. It is clear, then, that a corresponding parameter for a time-discrete prescription of ΔI_{n+1} as a function of ΔI_n is

$$\gamma \equiv \frac{1}{\Delta I_n}\frac{\Delta I_{n+1} - \Delta I_n}{\Delta t}$$

So the prescription becomes

$$\dot{I}(t_n)\Delta t \approx \Delta I_{n+1} = (1+\gamma\Delta t)\Delta I_n = (1+\gamma\Delta t)^2\Delta I_{n-1} = \cdots = (1+\gamma\Delta t)^n\Delta I_1 \quad (24)$$

It is clear that η and γ carry the same meaning in their respective contexts, and so γ is sensible inasmuch as η is sensible (See page 3).

Consideration of the $\dot{E}(t_n)$ term in (23) is analogous. So with $\Delta E_{n+1} \equiv E(t_{n+1}) - E(t_n)$, a prescription for ΔE_{n+1} as a function of E_n is

$$\dot{E}(t_n)\Delta t \approx \Delta E_{n+1} = (1+\delta\Delta t)\Delta E_n = (1+\delta\Delta t)^2\Delta E_{n-1} = \cdots = (1+\delta\Delta t)^n\Delta E_1$$

Here δ carries the same meaning in this time-discrete context as ε does in the time-continuous context (See page 4).

So, taking all together, the net monetary worth at time t_{n+1} may be approximated with

$$
\begin{aligned}
W(t_0) &= W_0 = S_0 + A_0 \qquad\qquad \text{(Equation (1))} \\
W(t_{n+1}) \approx W_{n+1} &= (1+\sigma\Delta t)W_n + (\alpha-\sigma)\Delta t A_n \\
&\qquad + (1+\gamma\Delta t)\Delta I_n \\
&\qquad - (1+\delta\Delta t)\Delta E_n \\
&= (1+\sigma\Delta t)W_n + (\alpha-\sigma)\Delta t(1+\alpha\Delta t)^n A_0 \\
&\qquad + (1+\gamma\Delta t)^n \Delta I_1 \\
&\qquad - (1+\delta\Delta t)^n \Delta E_1 \qquad\qquad \text{for } n = 0,1,2,\ldots
\end{aligned}
\tag{25}
$$

We now use this recursive discrete formula to derive a closed-form non-recursive formula and see how it compares to the corresponding formula for the time-continuous case. For temporary ease of algebraic manipulation, define

$$
a = 1+\delta\Delta t, \quad b = 1+\sigma\Delta t, \quad g = 1+\gamma\Delta t, \quad p = 1+\alpha\Delta t
$$

Then

$$
\begin{aligned}
W_1 &= bW_0 + (p-b)A_0 + \Delta I_1 - \Delta E_1 \\
&= b(S_0+A_0) + (p-b)A_0 + \Delta I_1 - \Delta E_1 \\
&= b^1 S_0 + p^1 A_0 + b^0 \sum_{j=0}^{0} \left(\frac{g}{b}\right)^j \Delta I_1 - b^0 \sum_{j=0}^{0} \left(\frac{a}{b}\right)^j \Delta E_1
\end{aligned}
$$

$$
\begin{aligned}
W_2 &= bW_1 + (p-b)pA_0 + g\Delta I_1 - a\Delta E_1 \\
&= b[bS_0 + pA_0 + \Delta I_1 - \Delta E_1] + (p-b)pA_0 + g\Delta I_1 - a\Delta E_1 \\
&= b^2 S_0 + p^2 A_0 + b^1 \sum_{j=0}^{1} \left(\frac{g}{b}\right)^j \Delta I_1 - b^1 \sum_{j=0}^{1} \left(\frac{a}{b}\right)^j \Delta E_1
\end{aligned}
$$

$$
W_3 = b^3 S_0 + p^3 A_0 + b^2 \sum_{j=0}^{2} \left(\frac{g}{b}\right)^j \Delta I_1 - b^2 \sum_{j=0}^{2} \left(\frac{a}{b}\right)^j \Delta E_1
$$

Observing the partial sum of the geometric series,

$$
\sum_{j=0}^{m} x^j = \frac{1-x^{m+1}}{1-x} \qquad \text{for any } x \text{ and for } m = 0,1,2,\ldots
$$

and continuing for subsequent discrete times, we obtain an expression for W_n at any time t_n as

$$
\boxed{
\begin{aligned}
W(t_n) \approx W_n = (1+\sigma\Delta t)^n S_0 &+ (1+\alpha\Delta t)^n A_0 \\
&+ \frac{(1+\sigma\Delta t)^n - (1+\gamma\Delta t)^n}{(\sigma-\gamma)\Delta t}\Delta I_1 \\
&- \frac{(1+\sigma\Delta t)^n - (1+\delta\Delta t)^n}{(\sigma-\delta)\Delta t}\Delta E_1 \qquad \text{for } n = 0,1,2,\ldots
\end{aligned}
}
\tag{26}
$$

The derivation of the time-continuous formula (13) for $W(t)$ involved the integration of differential equations. But the derivation here of the time-discontinuous formula (26) began with (25) and relied only on the well-known partial sum of a geometric series. It is therefore worth comparing the two cases, at least briefly, and checking for correspondence. Consider the term in (13) involving $\dot{I}(t_0)$ at $t = t_n$.

$$\begin{aligned}
\frac{\dot{I}(t_0)}{\eta - \sigma}\left(e^{\eta(t_n - t_0)} - e^{\sigma(t_n - t_0)}\right) &= \frac{\dot{I}(t_0)}{\eta - \sigma}\left(e^{n\eta\Delta t} - e^{n\sigma\Delta t}\right) \\
&= \frac{\dot{I}(t_0)}{\eta - \sigma}\left(1 + n\eta\Delta t - 1 - n\sigma\Delta t + o(\Delta t^2)\right) \qquad \text{(Taylor expansion)} \\
&= n\dot{I}(t_0)\Delta t + o(\Delta t^2) \\
&= n\Delta I_1 + o(\Delta t^2) \qquad \text{(Using (24))}
\end{aligned}$$

The corresponding term in (26) is

$$\begin{aligned}
\frac{(1 + \sigma\Delta t)^n - (1 + \gamma\Delta t)^n}{(\sigma - \gamma)\Delta t}\Delta I_1 &= \frac{1 + n\sigma\Delta t - 1 - n\gamma\Delta t + o(\Delta t^2)}{(\sigma - \gamma)\Delta t}\Delta I_1 \\
&= n\Delta I_1 + o(\Delta t)
\end{aligned}$$

which affirms the correspondence to order Δt.

4 Case studies

In this section, I present three case studies in which the financial modelling work described in Section 2 has been applied. In the first study, I examined the impact of retaining versus selling an income-generating asset. In the second study, I examined the impact of rising expense rates on an investor's present and future financial security. In the third study, I considered how present financial circumstance may relate to financial security during retirement.

4.1 Asset or savings

An individual investor owns: a primary residence that is free of any mortgage, a portfolio of equities, a positive balance in a moneymarket account, and a secondary residence that has hitherto been functioning as an income-generating guesthouse. While the guesthouse boosts average income, it also incurs additional expenses. Should the investor retain the guesthouse and enjoy the additional net income, or sell it and enjoy the additional interest arising from the extra savings? Forecasting the impact of the two scenarios on the investor's net monetary worth will, I believe, answer this question. Of course, this is assuming the investor is a prudent saver, as discussed in Subsection 2.2 on page 4.

Histories of projected net monetary worth W for the two scenarios were calculated using (13) on page 5, and are shown in Figure 1 below. Comparing the two histories clearly shows that given the investor's context, selling the guesthouse now is preferable.

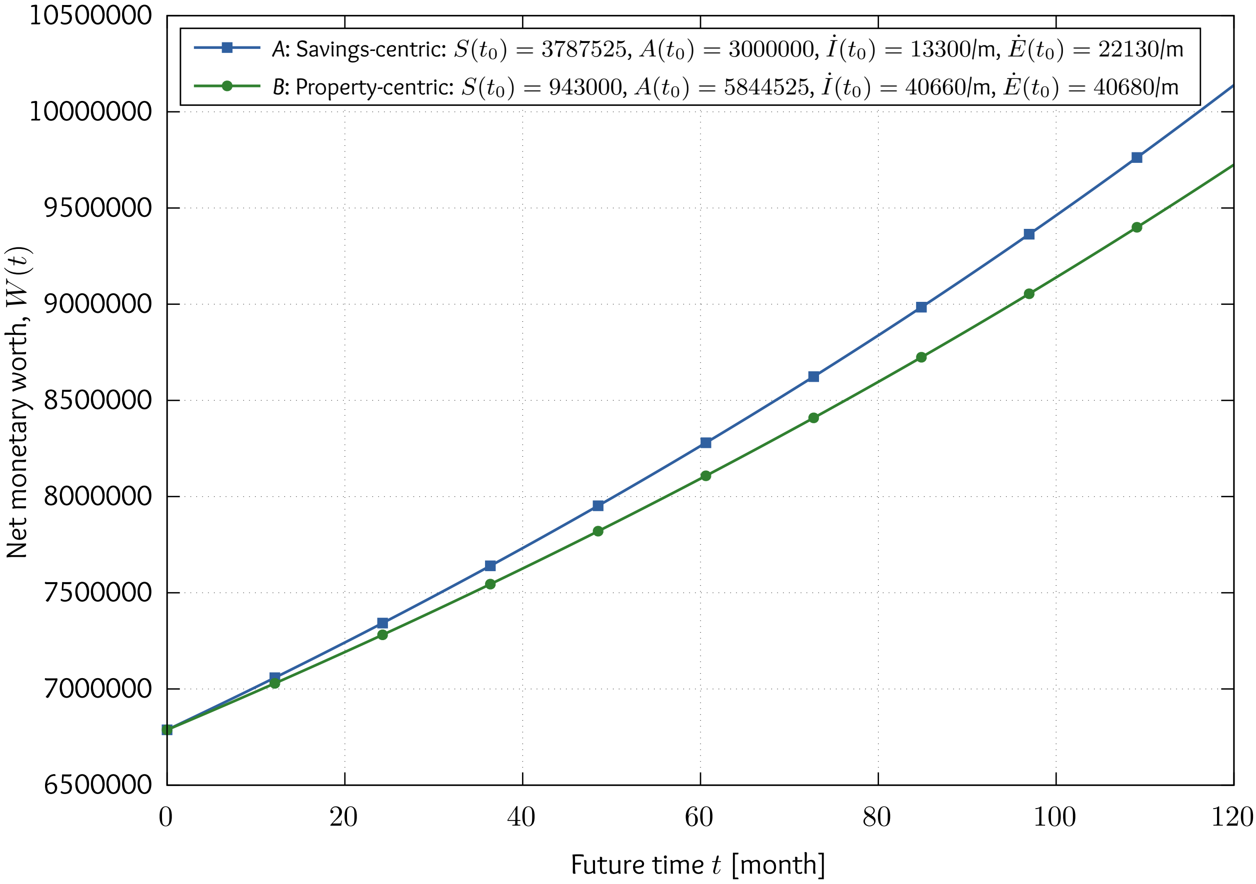

Figure 1: Two scenarios for projected net monetary worth, W, were calculated using (13) on page 5. In Scenario *A*, the investor owns a guesthouse but elects to sell it and use the proceeds to supplement savings. In Scenario *B*, the investor retains the guesthouse and enjoys income from it, but also incurs expenses pertaining to the guesthouse. The rate of relative increase in savings was set to $\sigma = 0.075/12$ per month. The rate of relative increase in the value of assets was set to $\alpha = 0.028/12$ per month. The rate of relative increase in receipt of income was set to $\eta = (0.035)/12$ per month. And the inflation rate was set to $\varepsilon = 0.035/12$ per month.

4.2 Rising expense rates

Financial security criterion. The invidivual investor mentioned above in Subsection 4.1 has now sold the secondary residence that had hitherto been functioning as an income-generating guesthouse. Indeed, the results in that subsection deemed the sale to be wise because it resulted in a greater net monetary worth. As a prudent saver, the investor now injects the proceeds of the sale into savings. Having done so, we now focus on the investor's overall financial security. To address this, we appeal to three calculated projected histories of the financial security criterion, C_{sec}, in (14) on page 6. The histories are plotted in Figure 2 below.

The differences in the history curves obtain from nothing more than differences in the rate at which expenses are incurred at the present time. The differences are striking and informative. With an expense rate of 22130 per month (the blue curve in Figure 2), the investor will always be financially secure because $C_{\text{sec}}(t; t_0) > 0$ for all future times $t \geqslant t_0$. However, with a higher expense rate of 24453.65 per month (the green curve in Figure 2), the investor starts out being financially insecure ($C_{\text{sec}}(t_0; t_0) < 0$), but becomes financially secure after about 46 months. This decline in financial security when transitioning from the blue curve to the green curve was caused by a mere 10.5% increase in expense rate. Finally, with an expense rate of 26556 per month (the red curve), the investor starts out being financially insecure ($C_{\text{sec}}(t_0; t_0) < 0$), and remains insecure forever. That is, for this investor, a transition from perpetual financial security (the blue curve) to perpetual financial insecurity (the red curve) was caused by a mere 20% increase in present expense rate!

Saving for financial security. The above financial security criterion, C_{sec}, is useful because it combines the present financial variables of savings $S(t_0)$, non-fungible assets $A(t_0)$, income rate $\dot{I}(t_0)$, and expense rate $\dot{E}(t_0)$ into a single encompassing relation. Some people prefer holding non-fungible assets such as property over tradeable investment instruments such as equities and interest-bearing cash accounts. The C_{sec} criterion does not explicitly prescribe holding one variable over the other, and so it is useful for all people regardless of their preferences.

However, this particular investor would like to concentrate on present savings $S(t_0)$ for now. The investor would like to know what his/her present savings should be in order to enjoy financial security either now or at some future specified time. To address this, we appeal to three calculated projected histories of a criterion for present savings. That criterion is S_{sec} from (15) on page 6. $S_{\text{sec}}(t; t_0)$ is a prescribed lower bound for $S(t_0)$ which must be honoured at the present time in order for the investor to be financially secure at a future specified time t. The projected histories of S_{sec} are plotted in Figure 3 on page 16. Note that the plots in Figures 2 and 3 actually convey the same information, just in two different ways. Both figures link the investor's present financial circumstance to future financial security, but the plots in Figure 3 focus on present savings.

With an expense rate of 22130 per month (the blue curve in Figure 3), the investor will always be financially secure because $S(t_0) \geqslant S_{\text{sec}}(t; t_0)$ for all times $t \geqslant t_0$. That is, the purple line in Figure 3 remains above the blue curve for all times. However, with an expense rate of 24453.65 per month (the green curve), the investor starts out being financially insecure because $S(t_0) < S_{\text{sec}}(t_0; t_0)$, but becomes financially secure after about 46 months. That is, the green curve starts out above the purple line but dips below after about 46 months. Comparing the blue curve with the red curve, an increase in expense rate from 22130 per month to 26556 per month pushes up the lower bound for present savings from 3174000 to 4501800 per month, an increase of 1327800 per month. In other words, a 20% increase in the present expense rate pushes up the lower bound for present savings by 41%. So once again, the investor's expense rate is shown to be a powerful lever.

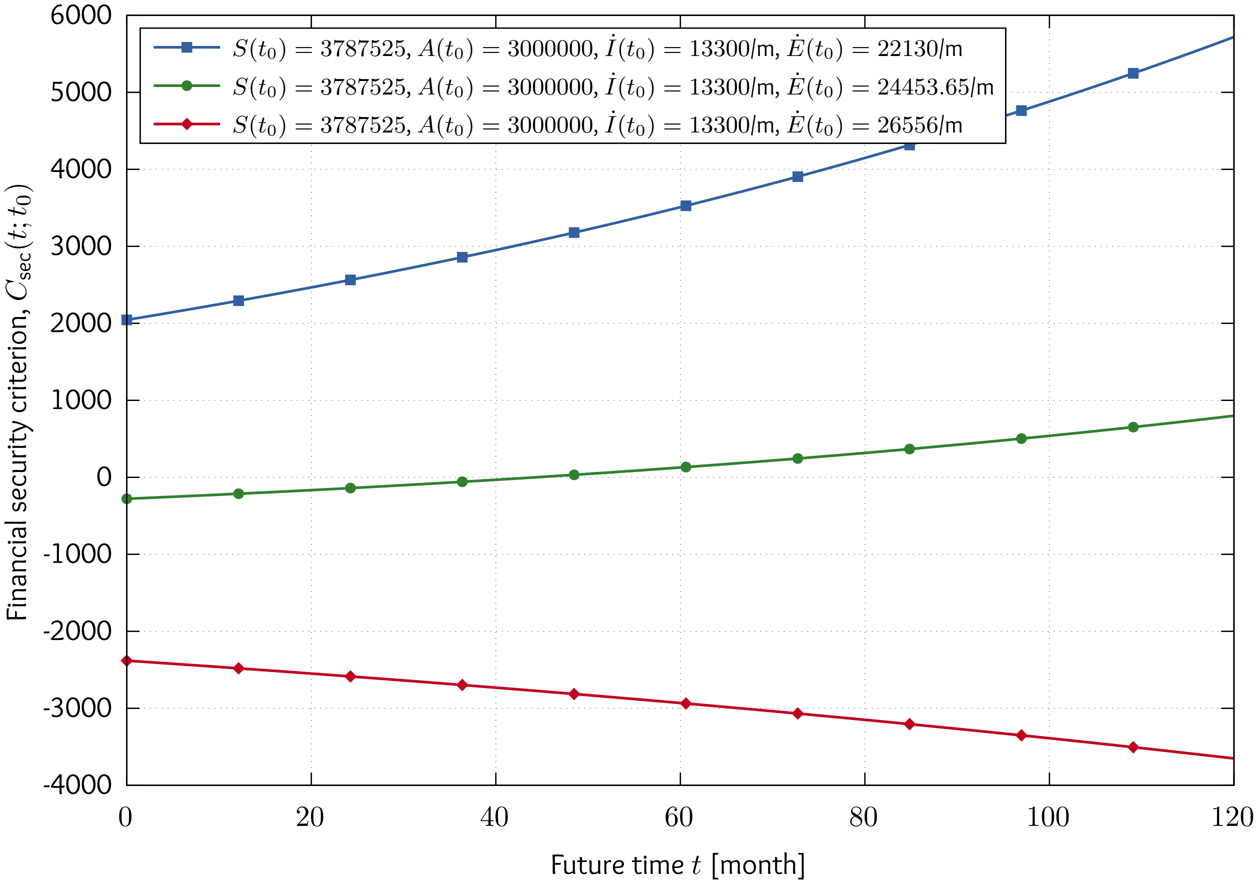

Figure 2: Financial security criterion, C_{sec}, calculated for three scenarios. The only difference between each scenario is the present rate, $\dot{E}(t_0)$, at which expenses are incurred. $C_{\text{sec}}(t; t_0)$ was calculated using (14) on page 6. The investor is deemed financially secure whenever $C_{\text{sec}} \geqslant 0$. The three histories taken together demonstrate the sensitivity for this investor's financial security to relatively small increases in the rate of present expenses, $\dot{E}(t_0)$. By increasing the expense rate from 22130 per month to 26556 per month, a mere increase of **4426** per month, the investor loses present financial security. Forever. The required input parameters were obtained from a client's real life circumstance. The rate of relative increase in savings was set to $\sigma = 0.075/12$ per month. The rate of relative increase in the value of assets was set to $\alpha = 0.028/12$ per month. The rate of relative increase in receipt of income was set to $\eta = (0.035)/12$ per month. And the inflation rate was set to $\varepsilon = 0.035/12$ per month.

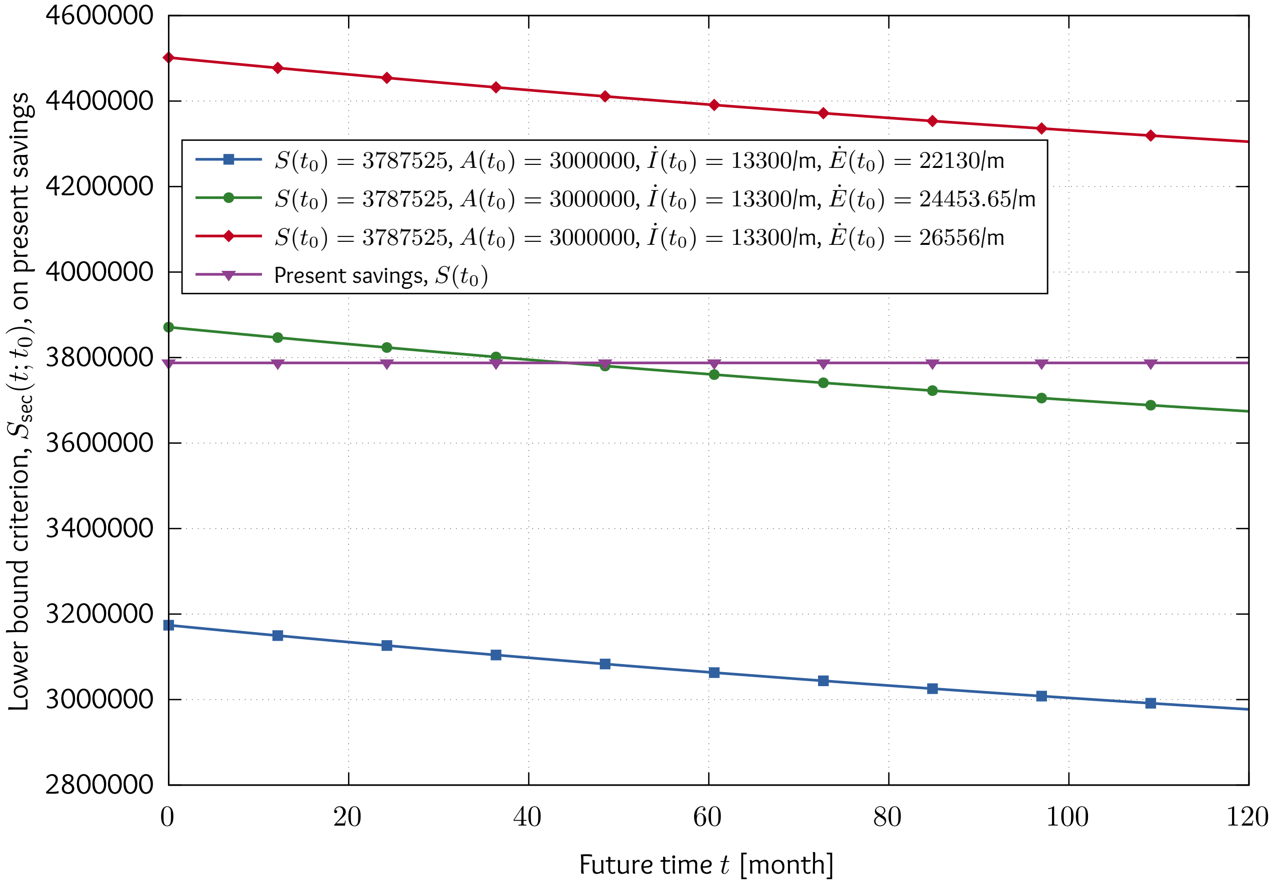

Figure 3: Lower bound criterion, S_{sec}, for present savings, $S(t_0)$. $S_{\text{sec}}(t;t_0)$ was calculated using (15) on page 6. This criterion answers the question, "What must the investor's present savings be in order for him/her to enjoy financial security at some future specified time?" And its answer is, simply, $S(t_0) \geqslant S_{\text{sec}}(t;t_0)$ for the specified time t. The required input parameters were obtained from a client's real life circumstance. The rate of relative increase in savings was set to $\sigma = 0.075/12$ per month. The rate of relative increase in the value of assets was set to $\alpha = 0.028/12$ per month. The rate of relative increase in receipt of income was set to $\eta = (0.035)/12$ per month. And the inflation rate was set to $\varepsilon = 0.035/12$ per month.

4.3 Retirement

An individual investor presently enjoys a labour-based income stream, and has been contributing to a pension fund from which he/she expects to obtain an income stream during retirement, albeit at a reduced rate. The investor wishes to retire in the near future, say 60 months from now. Will his/her present financial circumstance enable retirement to be a financially secure one? Forecasting the impact of the event of retirement, which might accompany an abrupt change in income from labour, will, I believe, answer this question. Once again, it is assumed that the investor is a prudent saver, as discussed in Subsection 2.2 on page 4.

Net monetary worth pre- and post-retirement. A history of the investor's projected net monetary worth W from now until the time retirement, $t_{\mathsf{R}} = 60$ months, was calculated using (13) on page 5. Next, a set of histories of projected net monetary worth W_{R} during of retirement, $t \geqslant t_{\mathsf{R}}$, was calculated using (19) on page 8. All the histories are shown in Figure 4 below.

The impact on monetary net worth of the abrupt transition from a labour-based income to a pension income at time t_{R} is clearly evident. But it will require deeper analyses to appreciate the seriousness of that impact (See below). Note that, as expected, with no change in the income stream during retirement ($\lambda = 1$), the projected net monetary worth during retirement, namely, the green curve in Figure 4, coincides with the "Savings-centric" curve in Figure 1 on page 13.

Also, as expected, abruptly changing the investor's income stream at the onset of retirement at time t_{R} results in lower projected net monetary worths during retirement. Although the worths are lower, they are still rising. But are they rising sufficiently for the investor to be financially secure during retirement? To answer, we must appeal to a financial security criterion.

Financial security pre- and post-retirement. The three calculated projected histories of the financial security criterion C_{sec} (Equation (14) on page 6), and which are plotted in Figure 2 on page 15, do not account for an abrupt change in an investor's financial circumstance, such as is often the case when a person retires. To account for such an abrupt change, we must appeal to two criteria, namely, to C_{sec} for times prior to the change, and to C_{Rsec} for times after it. A projected history for the C_{sec} criterion for all future times prior to retirement ($t_0 \leqslant t < t_{\mathsf{R}}$) was calculated using (14) on page 6. And a projected history for the C_{Rsec} criterion for all future times during retirement ($t \geqslant t_{\mathsf{R}}$) was calculated using (20) on page 9. The histories are plotted in Figure 5 page 19.

With the investor's present financial circumstance characterised by the combination of values for $S(t_0)$, $A(t_0)$, $\dot{I}(t_0)$ and $\dot{E}(t_0)$, it is projected that the investor will remain financially secure prior to retirement ($C_{\mathsf{sec}} \geqslant 0$) and during retirement ($C_{\mathsf{Rsec}} \geqslant 0$), provided that his/her pension income remains roughly on par with his/her labour-based income. That is, both the blue curve and the green curve remain positive for all times ($C_{\mathsf{sec}}(t; t_0) \geqslant 0$ and $C_{\mathsf{Rsec}}(t; t_0) \geqslant 0$ for all t).

However, it is noteworthy that with this investor's present financial circumstance and with pension income at the onset of retirement being a fraction $\lambda = 0.75$ of labour-based income, the investor will have to wait no less than 60 months into retirement to regain financial security. This is shown by the red curve in Figure 5. Whereas, if at the onset of retirement, pension income is expected to be a fraction $\lambda = 0.5$ of labour-based income, then the investor's retirement will never be a financially secure one. This is shown by the purple curve in Figure 5 always being negative, and strongly decreasing.

What can we say about the investor's present savings? Projected histories of the savings criteria S_{sec} and S_{Rsec} were calculated using (15) on page 6 and (21) on page 9. S_{sec} is the prescribed lower bound on present savings $S(t_0)$ in order for financial security before retirement ($t_0 \leqslant t < t_{\mathsf{R}}$), and S_{Rsec} is the same but for financial security during retirement ($t \geqslant t_{\mathsf{R}}$). The histories plotted in Figure 6 on page 20 tell the same story as above. Namely, if the pension income at the onset of retirement is anything less than the fraction $\lambda = 0.75$ of labour-based income, then the investor's present savings are not sufficient for a financially secure retirement. That's because the investor's present savings (the brown line in Figure 6) remains below the threshold prescribed by S_{Rsec} during retirement (the red curve and the purple curve).

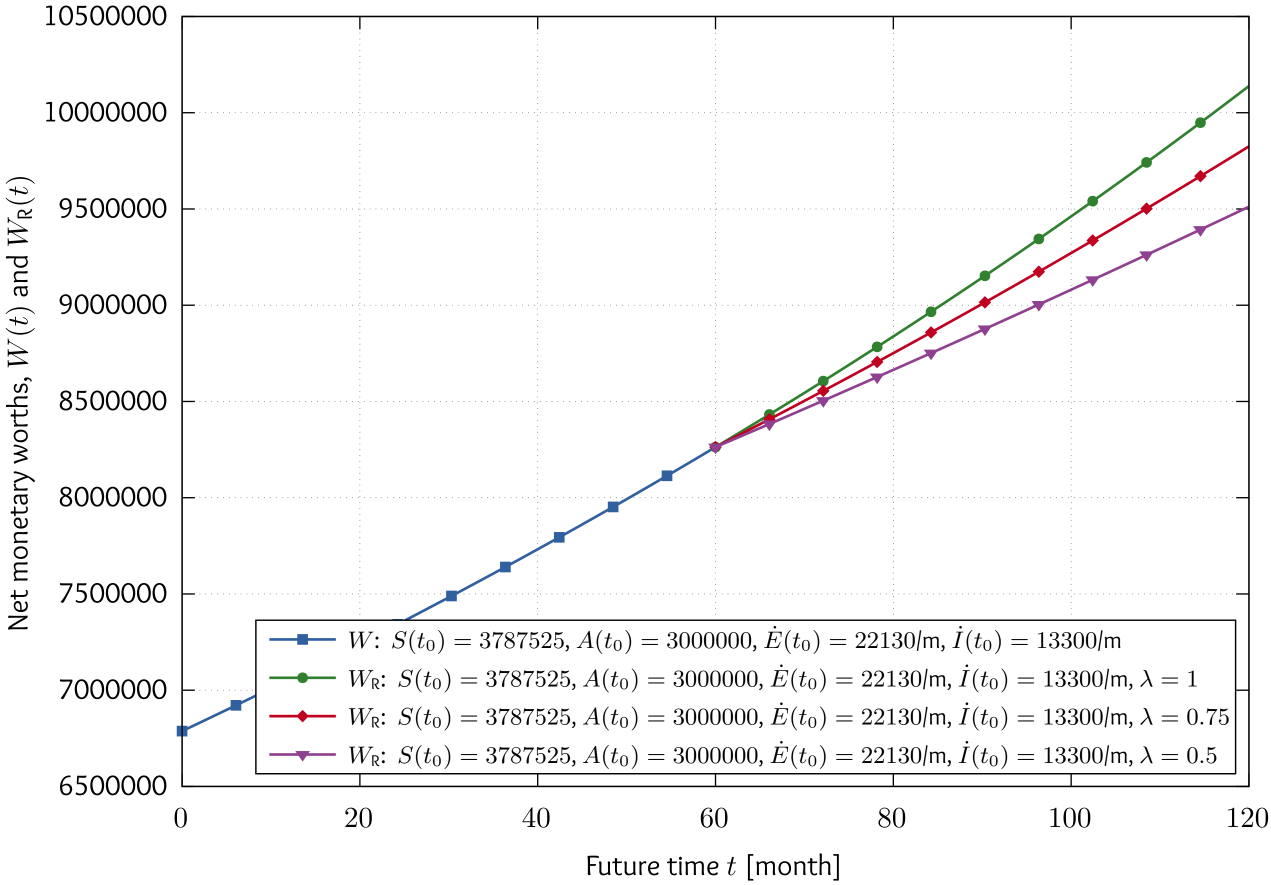

Figure 4: Projected net monetary worth, W, before retirement, and during retirement, W_R. $W(t)$ was calculated using (13) on page 5, and $W_R(t)$ was calculated using (19) on page 8. The time of onset of retirement was set to $t_R = 60$ months. Abrupt changes in the investor's projected income stream at the onset of retirement were modelled by setting $\lambda = 1$, $\lambda = 0.75$ and $\lambda = 0.5$. Recall that λ is defined in (18) on page 8 as the ratio of pension income to labour-based income at time t_R, namely, $\dot{I}_R(t_R)/\dot{I}(t_R)$. The effect on the net monetary worth during retirement is clearly evident. The rate of relative increase in savings was set to $\sigma = 0.075/12$ per month. The rate of relative increase in the value of assets was set to $\alpha = 0.028/12$ per month. The rate of relative increase in receipt of income was set to $\eta = (0.035)/12$ per month. And the inflation rate was set to $\varepsilon = 0.035/12$ per month.

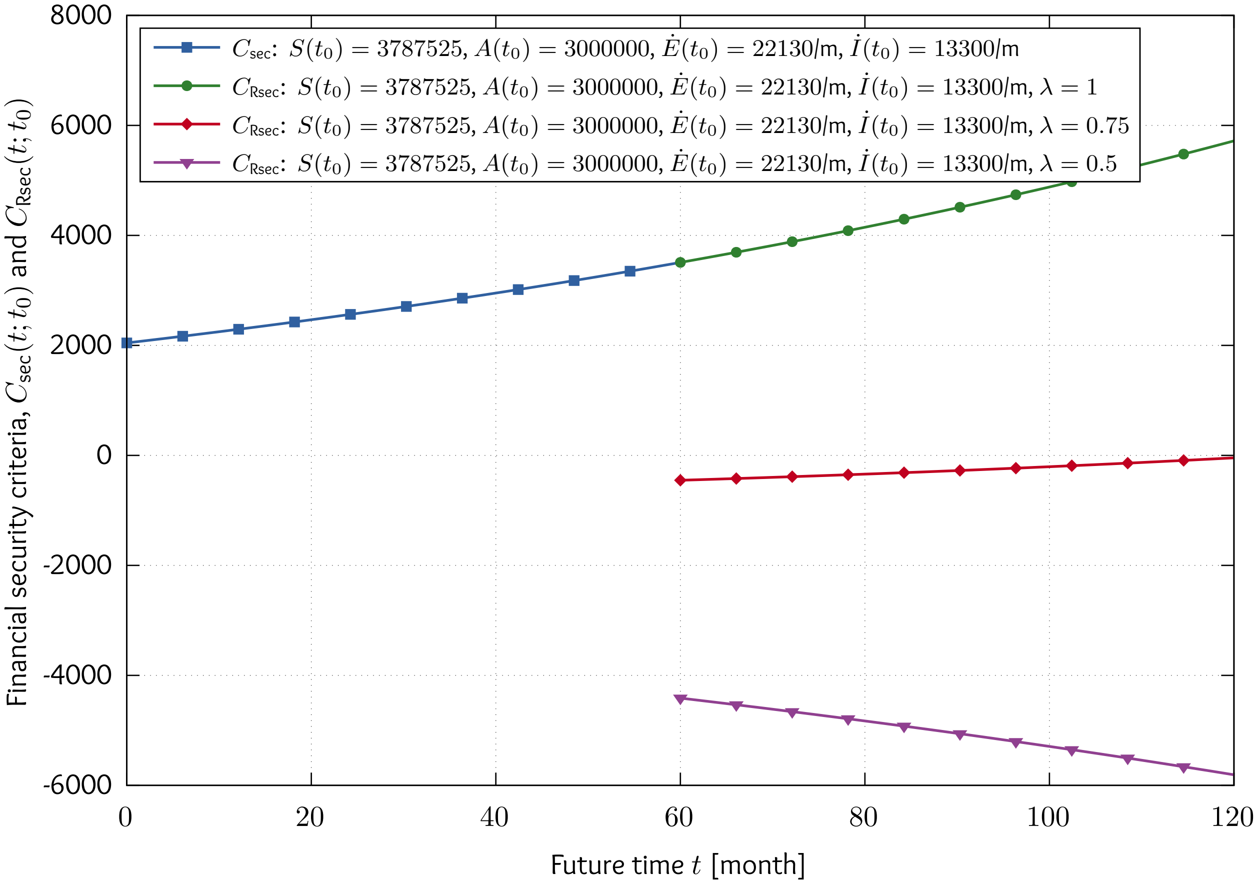

Figure 5: Financial security criteria before retirement, C_{sec}, and during retirement, C_{Rsec}. $C_{\mathsf{sec}}(t; t_0)$ was calculated using (14) on page 6, and $C_{\mathsf{Rsec}}(t; t_0)$ was calculated using (20) on page 9. The investor is deemed financially secure whenever $C_{\mathsf{sec}} \geqslant 0$ or $C_{\mathsf{Rsec}} \geqslant 0$. The time of onset of retirement was set to $t_{\mathsf{R}} = 60$ months. Abrupt changes in the investor's projected income stream at the onset of retirement were modelled by setting $\lambda = 1$, $\lambda = 0.75$ and $\lambda = 0.5$. Recall that λ is defined in (18) on page 8 as the ratio of pension income to labour-based income at time t_{R}, namely, $\dot{I}_{\mathsf{R}}(t_{\mathsf{R}})/\dot{I}(t_{\mathsf{R}})$. The rate of relative increase in savings was set to $\sigma = 0.075/12$ per month. The rate of relative increase in the value of assets was set to $\alpha = 0.028/12$ per month. The rate of relative increase in receipt of income was set to $\eta = (0.035)/12$ per month. And the inflation rate was set to $\varepsilon = 0.035/12$ per month.

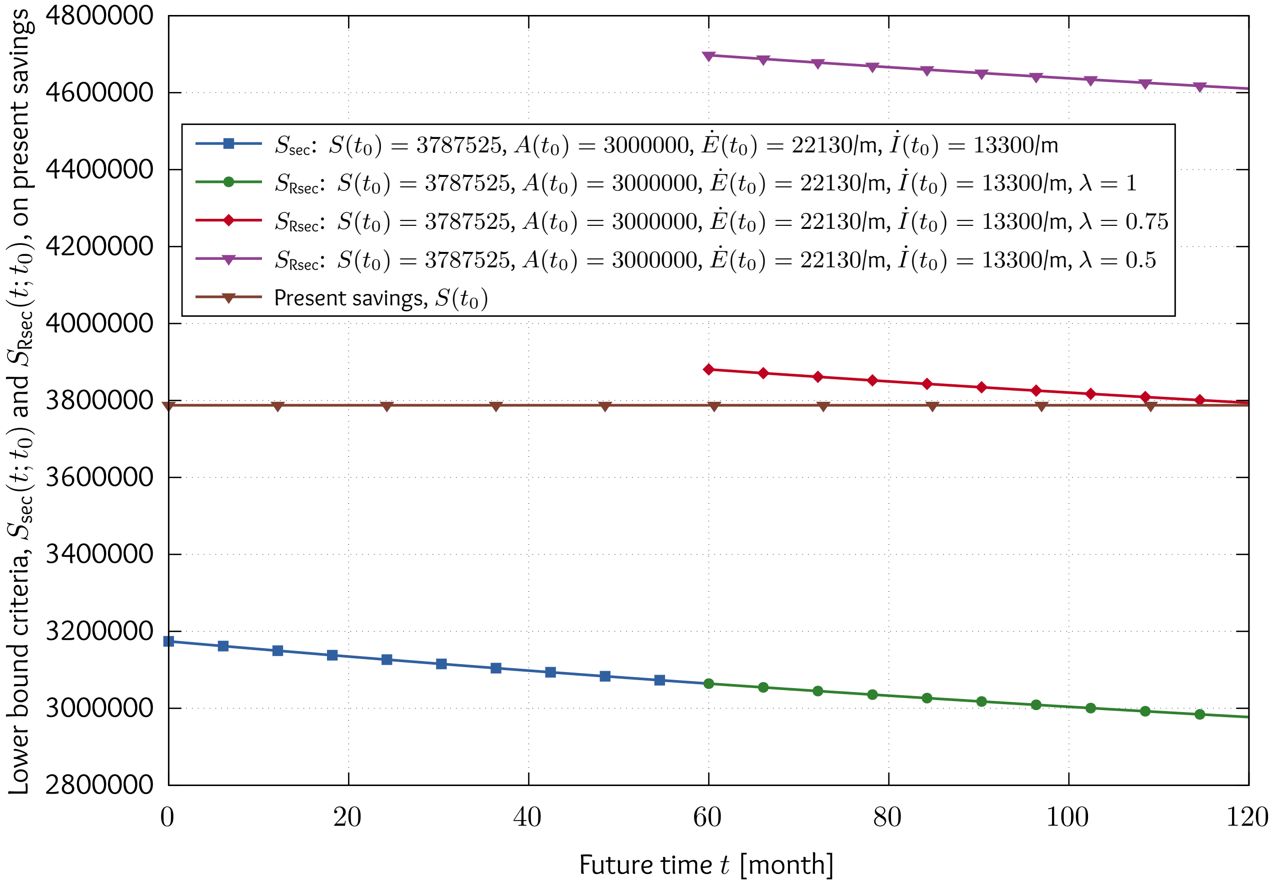

Figure 6: Lower bound criteria, S_{sec} and S_{Rsec}, for present savings, $S(t_0)$. $S_{\text{sec}}(t;t_0)$ was calculated using (15) on page 6, and $S_{\text{Rsec}}(t;t_0)$ was calculated using (21) on page 9. The S_{sec} criterion answers the question, "What must the investor's present savings be in order for him/her to enjoy financial security at some future specified time but prior to retirement?" And its answer is, simply, $S(t_0) \geqslant S_{\text{sec}}(t;t_0)$. The S_{Rsec} criterion answers the same question but for some future specified time during retirement. And its answer is, $S(t_0) \geqslant S_{\text{Rsec}}(t;t_0)$. The time of onset of retirement was set to $t_{\text{R}} = 60$ months. Abrupt changes in the investor's projected income stream at the onset of retirement were modelled by setting $\lambda = 1$, $\lambda = 0.75$ and $\lambda = 0.5$. Recall that λ is defined in (18) on page 8 as the ratio of pension income to labour-based income at time t_{R}, namely, $\dot{I}_{\text{R}}(t_{\text{R}})/\dot{I}(t_{\text{R}})$. The rate of relative increase in savings was set to $\sigma = 0.075/12$ per month. The rate of relative increase in the value of assets was set to $\alpha = 0.028/12$ per month. The rate of relative increase in receipt of income was set to $\eta = (0.035)/12$ per month. And the inflation rate was set to $\varepsilon = 0.035/12$ per month.

Spending for retirement. The above data do not inspire confidence that the investor's present financial circumstance will enable a financially secure retirement. So what can be done? For one, the investor could try to reduce his/her present rate $E(t_0)$ at which expenses are incurred.

Projected histories of the financial criteria C_{sec} and C_{Rsec} were calculated for two present financial circumstances for the investor. In both circumstances, it was assumed that at the planned onset of retirement in $t_{\text{R}} = 60$ months time, the investor will suffer an abrupt 50% drop ($\lambda = 0.5$) in his/her income stream. These histories are shown in Figure 7 below.

In the first circumstance, the investor's expense rate is 22130 per month (the blue curve and the green curve). This matches the histories shown in Figure 5 on page 19. In the second circumstance, the expense rate is reduced to 18810.5 per month (the red curve and the purple curve). By reducing the present expense rate by this relatively small amount—a mere 15%—the investor regains the prospect of future financial security during retirement. This is shown in Figure 7 by the purple curve being positive for all times during retirement, and the the green curve being negative.

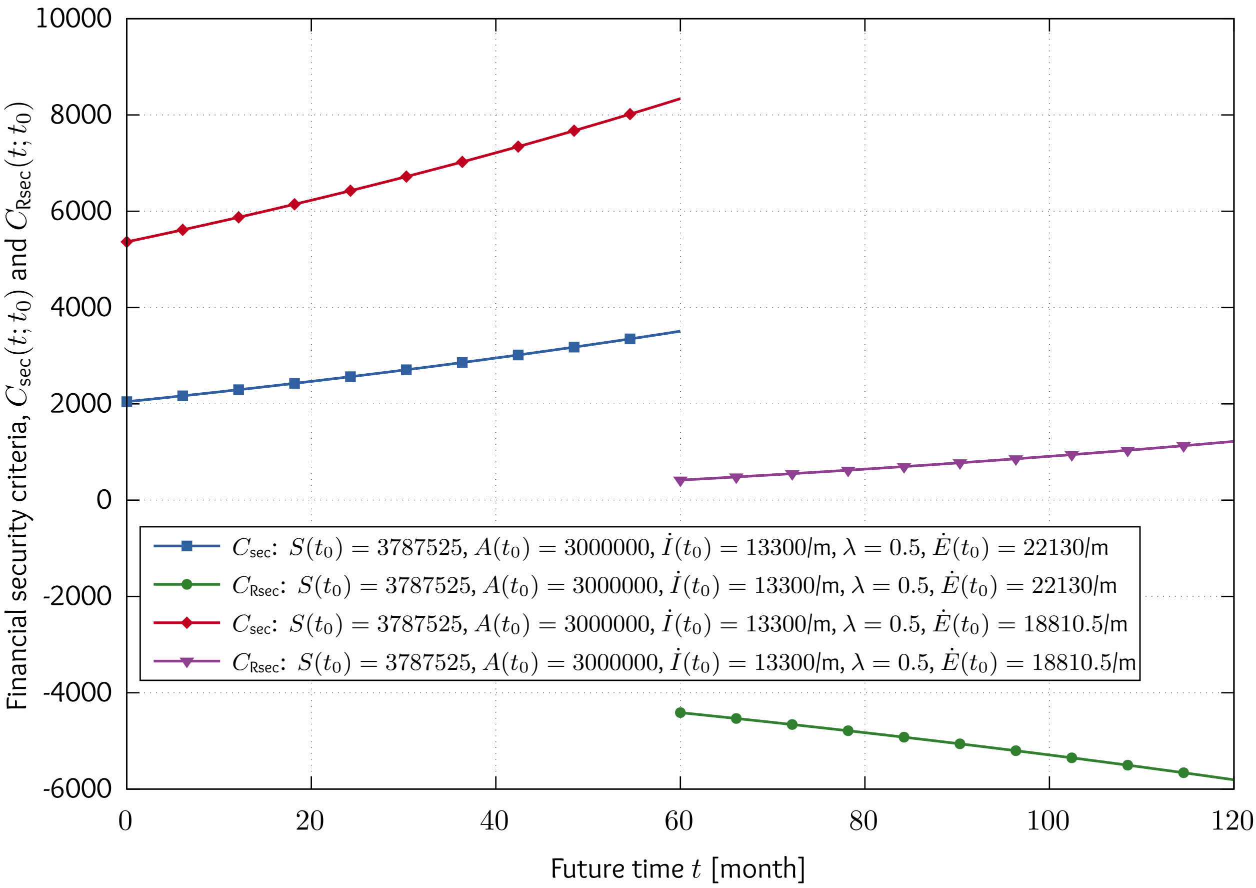

Figure 7: Financial security criteria for future times before retirement and during retirement. C_{sec} is for times before, and C_{Rsec} is times during retirement. $C_{\text{sec}}(t;t_0)$ was calculated using (14) on page 6, and $C_{\text{Rsec}}(t;t_0)$ was calculated using (20) on page 9. The investor is deemed financially secure whenever $C_{\text{sec}} \geqslant 0$ or $C_{\text{Rsec}} \geqslant 0$. The time of onset of retirement was set to $t_{\text{R}} = 60$ months. An abrupt change in the investor's projected income stream at the onset of retirement was modelled by setting $\lambda = 0.5$. Recall that λ is defined in (18) on page 8 as the ratio of pension income to labour-based income at time t_{R}, namely, $\dot{I}_{\text{R}}(t_{\text{R}})/\dot{I}(t_{\text{R}})$. Taken together, the histories demonstrate the importance for this investor of reducing present expenses. By reducing the expense rate from **22130** per month to **18810.5** per month, the investor regains the prospect of future financial security during retirement. This is shown by the purple curve being positive for all times during retirement, and the the green curve being negative. The rate of relative increase in savings was set to $\sigma = 0.075/12$ per month. The rate of relative increase in the value of assets was set to $\alpha = 0.028/12$ per month. The rate of relative increase in receipt of income was set to $\eta = (0.035)/12$ per month. And the inflation rate was set to $\varepsilon = 0.035/12$ per month.

5 Acknowledgments

As always Mels, thanks for being such a close friend and supportive partner. Your worth to me is incalculable and immeasurable.